Several Remarks on Infinite Series

Comment # 72 from Enestroemiani's Index
Commentarii academiae scientarum Petropolitanae **9** (1737), **1744**, p. 160–188

Leonard Euler

The remarks I have decided to present here refer generally to that kind of series which are absolutely different from the ones usually considered till now.

But in the same way that, up to date, the only series which have been considered are those whose general terms are given or, at least, the laws under which, given a few terms the rest can be found are known, I will here consider mainly those series that have neither a general term as such nor a continuation law but whose nature is determined by other conditions.

Thus, the most amazing feature of this kind of series would be the possibility of summing them up, as the known methods till now require necessarily the general term or the continuation law without which it seems obvious that we cannot find any other means of obtaining their sums.

I was prompted to these remarks by a special series communicated to me by Cel. Goldbach[1] whose astonishing sum, with the liege of the Celebrated Master, I hereby present in the first place.

Theorem 1. *Consider the following series, infinitely continued,*

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\frac{1}{31}+\frac{1}{35}+\cdots$$

whose denominators, increased by one, are all the numbers which are powers of the integers, either squares or any other higher degree. Thus each term may be expressed by the formula $\dfrac{1}{m^n-1}$ *where m and n are integers greater than one. The sum of this series is 1.*

Proof. This is the first Theorem that Celeb. Goldbach communicated to me and which prompted me to make the statements that follow. For from the close inspection of this series it is seen the irregularity that it shows in its progression and thus anyone versed in these questions will marvel at how the Cel. Master discovered the sum of this singular series; and this is the way in which he proved it to me.

Let

$$x=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots;$$

from here, as we have

$$1=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots,$$

it will result, subtracting this series from the former

$$x-1=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10}\cdots;$$

[1] Goldbach's letter here mentioned by Euler is not extant but related to it there is the letter from Goldbach to Daniel Bernoulli of April 1729, *Correspondance math. et phys. publiée par P. H. FUSS*, St.–Pétersbourg 1843, t. II, p. 296. See also the *postscriptum* from 31 January, *ibid.*, and also the letter of 26 May 1729, *ibid.*, p. 283 and 305.

C.B.

thus all powers of two, including two itself, disappear from the denominators remaining all the other numbers.

Also, if from that series above we subtract this one

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$$

there will result

$$x - 1 - \frac{1}{2} = 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \cdots ;$$

and subtracting again

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$$

it will remain

$$x - 1 - \frac{1}{2} - \frac{1}{4} = 1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \cdots .$$

Proceeding similarly deleting all the terms that remain, we get finally

$$x - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{9} - \cdots = 1$$

or

$$x - 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \cdots$$

whose denominators, increased by one, are all the numbers which are not powers. Consequently, if we subtract this series from the series we have considered at the beginning

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots ;$$

we get

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \cdots ,$$

series whose denominators, increased by one, are all the powers of the integers and whose sum is one. Q. E. D.

Theorem 2. *The series, continued infinitely,*

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35} + \frac{1}{63} + \cdots$$

where the denominators, increased by one are all the even powers, has sum $l2$, and this series

$$\frac{1}{8} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \frac{1}{80} + \cdots ,$$

continued infinitely, whose denominators, increased by one are all the odd powers, has sum $1 - l2$. The former series has as general term $\frac{1}{(2m-2)^n - 1}$, and the latter responds to $\frac{1}{(2m-1)^n - 1}$, where m and n retain their previous values.

Proof. Let us consider the following series whose sum is x,

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \cdots .$$

Now, as

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

subtracting this series from the former, we get the following

$$x - 1 = \frac{1}{6} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{18} + \cdots ,$$

from which, subtracting

$$\frac{1}{5} = \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \cdots$$

it results

$$x - 1 - \frac{1}{5} = \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{18} + \cdots ;$$

in a similar way, from

$$\frac{1}{9} = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

we get

$$x - 1 - \frac{1}{5} - \frac{1}{9} = \frac{1}{12} + \frac{1}{14} + \frac{1}{18} + \cdots .$$

Subtracting in this way all the terms we get

$$x = 1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \cdots$$

whose denominators constitute the natural series of odd numbers excepting those that, increased by one are powers; and this can be seen from the progression of this series. But as we have

$$l2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

and also

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$$

it will result

$$x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots - l2.$$

Thus, subtracting from here the value of x we had previously found, we have

$$0 = \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35} + \cdots - l2$$

and thus

$$l2 = \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35} + \cdots ,$$

series whose denominators are those odd numbers such that, increased by one, are all the even powers. Consequently, the sum of this series is $l2$ as we had contended in the statement of the Theorem. Q. E. D.(the first statement).

Now, as from the previous Theorem we have

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \cdots$$

where the denominators, increased by one, are all the numbers that are powers, even or odd, we will have, subtracting the previous series from this one,

$$1 - l2 = \frac{1}{8} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \cdots$$

whose denominators are precisely those odd numbers that, increased by one, are all the odd powers. Q. E. D.

Theorem 3. *Let π be the perimeter of the circle whose diameter is one. We have*

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} - \frac{1}{120} - \frac{1}{124} - \frac{1}{168} - \frac{1}{224} + \frac{1}{244} - \frac{1}{288} - \cdots$$

series whose denominators are those numbers which are at the same time even–even[2] *and one unity greater or less than a power of an odd number. Those fractions whose denominators exceed a power by a unit will have a + sign; the rest a − sign.*

Proof. We know that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \cdots$$

series where the fractions with an even–even denominator minus one have a − sign and the rest a + sign. If to this series we add the geometric series

$$\frac{1}{4} = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots$$

we get

$$\frac{\pi}{4} + \frac{1}{4} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots,$$

and subtracting from this last one

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$$

we have

$$\frac{\pi}{4} + \frac{1}{4} - \frac{1}{4} = 1 - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots,$$

series where neither 3 nor 5 nor any of their powers is present. In a similar way we remove 7 and its powers by adding this series

$$\frac{1}{8} = \frac{1}{7} - \frac{1}{49} + \cdots,$$

and we have

$$\frac{\pi}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \cdots.$$

In the same way we remove those terms which are not powers (at the same time we remove the powers). Finally we will have

$$\frac{\pi}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{8} + \frac{1}{12} - \frac{1}{12} + \frac{1}{16} - \frac{1}{16} + \frac{1}{20} - \frac{1}{20} + \frac{1}{24} - \frac{1}{28} + \cdots = 1$$

or

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} - \frac{1}{120} - \cdots$$

as the repeated terms cancel themselves and only the solitary ones remain; for the solitary ones were fractions whose denominators are even–even numbers such that increased or decreased by one are powers of odd numbers. The signs of these terms abide by the prescribed law. Q. E. D.

Theorem 4. *If, as before, π denotes the perimeter of the circle whose diameter is one we have*

$$\frac{\pi}{4} - \frac{3}{4} = \frac{1}{28} - \frac{1}{124} + \frac{1}{244} + \frac{1}{344} + \cdots$$

series whose denominators are all even–even numbers and at the same time are non–square powers of odd numbers plus or minus unity. Those fractions whose denominator exceed such powers have a + sign; the rest, whose denominators are deficient by one unit from a non–square power, have a − sign.

[2] Even among the even, i.e., multiples of 4. We will refer to them as even–even in the following. In a similar way those even numbers which are not divisible by four will be the odd–even numbers.

P. V. *et altri.*

Proof. By Theorem 3 above, we have

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} - \frac{1}{120} - \frac{1}{124} - \cdots$$

series in which, at the beginning, there are denominators which are deficient by one unit from odd squares; and these fractions have all a $-$ sign. But as we have

$$\frac{1}{8} + \frac{1}{24} + \frac{1}{48} + \frac{1}{80} + \frac{1}{120} + \frac{1}{168} + \cdots = \frac{1}{4}$$

we will have, replacing this value of $1/4$ in all places where it appears,

$$\frac{\pi}{4} = 1 - \frac{1}{4} + \frac{1}{28} - \frac{1}{124} + \frac{1}{244} + \frac{1}{344} + \cdots$$

or

$$\frac{\pi}{4} - \frac{3}{4} = \frac{1}{28} - \frac{1}{124} + \frac{1}{244} + \frac{1}{344} + \cdots$$

series whose denominators are even–even numbers which are non–square powers of odd numbers (for the squares are already excluded) plus or minus one; and according to the excess or defect, the fractions have sign $+$ or $-$. Q. E. D.

Corollary 1. *In order to continue the series of all odd numbers which are not powers, we must take the powers of odd exponents and increase them or decrease them by one, for thus even–even numbers will also appear, and these will be the denominators of the series, abiding by the law of signs.*

Corollary 2. *In the same way that odd numbers (either of the form $4m-1$ or $4m+1$) the powers with odd exponents of the form $4m-1$, increased by one and those with exponents of the form $4m+1$, decreased by one, are also even–even numbers, and we will be able to equate $\frac{\pi}{4} - \frac{3}{4}$ to the series whose terms obey to the formula $\frac{1}{(4m-1)^{2n+1}+1}$ after being subtracted the terms that obey to the formula $\frac{1}{(4m+1)^{2n+1}-1}$, where m and n have to be replaced by all the positive integers, except those for which either $4m+1$ or $4m-1$ are powers.*

Corollary 3. *Thus, $\frac{\pi}{4} - \frac{3}{4}$ will equate to the following set of infinite series*

$$\frac{\pi}{4} - \frac{3}{4} = \begin{cases} \frac{1}{3^3+1} + \frac{1}{3^5+1} + \frac{1}{3^7+1} + \frac{1}{3^9+1} + \cdots \\ -\frac{1}{5^3-1} - \frac{1}{5^5-1} - \frac{1}{5^7-1} - \frac{1}{5^9-1} - \cdots \\ +\frac{1}{7^3+1} + \frac{1}{7^5+1} + \frac{1}{7^7+1} + \frac{1}{7^9+1} + \cdots \\ +\frac{1}{11^3+1} + \frac{1}{11^5+1} + \frac{1}{11^7+1} + \frac{1}{11^9+1} + \cdots \\ -\frac{1}{13^3-1} - \frac{1}{13^5-1} - \frac{1}{13^7-1} - \frac{1}{13^9-1} - \cdots \\ +\frac{1}{15^3+1} + \frac{1}{15^5+1} + \frac{1}{15^7+1} + \frac{1}{15^9+1} + \cdots \\ \qquad\qquad \cdots \end{cases}$$

Corollary 4. *Consequently, if we continue this series until its denominators become greater than* 100 000*, we will have*

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{28} - \frac{1}{124} + \frac{1}{244} + \frac{1}{344} + \frac{1}{1332} + \frac{1}{2188} - \frac{1}{2196} - \frac{1}{3124} + \frac{1}{3376} - \frac{1}{4912} + \frac{1}{6860} - \frac{1}{9260} + \frac{1}{12168} + \frac{1}{16808} + \frac{1}{19684} - \frac{1}{24388} + \frac{1}{29792} - \frac{1}{35936} + \frac{1}{42876} - \frac{1}{50652} + \frac{1}{59320} - \frac{1}{68920} - \frac{1}{78124} + \frac{1}{779508} - \frac{1}{91124}[\cdots].$$[3]

Corollary 5. *As all the denominators are divisible by 4, we have*

$$\pi = 3 + \frac{1}{7} - \frac{1}{31} + \frac{1}{61} + \frac{1}{86} + \frac{1}{333} + \frac{1}{547} - \frac{1}{549} - \frac{1}{781} + \frac{1}{844} - \cdots .$$

It is worth remarking that the two first terms constitute Archimedes' ratio of the perimeter of the circle to the diameter.

Theorem 5. *Retaining the previous meaning for* π *we have,*

$$\frac{\pi}{4} - l2 = \underbrace{\frac{1}{26} + \frac{1}{28}} + \underbrace{\frac{1}{242} + \frac{1}{244}} + \underbrace{\frac{1}{342} + \frac{1}{344}} + \cdots ,$$

series whose law is that the mid–value between the paired denominators whose difference is two, i.e. 27, 243, 343, ..., are powers with odd exponent of odd numbers which increased by one are either divisible by four or even–even numbers.[4]

Proof. Theorem 3 says

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} - \cdots ,$$

(fractions affected of the $-$ sign have denominators which are even–even numbers deficient by one from a power of an odd number whilst the fractions affected of the sign $+$ have denominators that which are also even–even numbers exceeding by one a power of an odd number), and we also have by Theorem 2,

$$1 - l2 = \frac{1}{8} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \frac{1}{80} + \cdots ,$$

series whose denominators are deficient by one from all powers of odd numbers and thus comprising all terms affected of the $-$ sign and further all the fractions whose denominators are odd–even numbers deficient by one from a power of an odd number. Consequently, if to this series we add the other one, we have

$$\frac{\pi}{4} - l2 = \frac{1}{26} + \frac{1}{28} + \frac{1}{242} + \frac{1}{244} + \frac{1}{342} + \frac{1}{344} + \cdots$$

the fractions of which are paired in twos in such way that the denominator of the first one of the two is an odd–even number and the second of the two is an even–even number and the mid–number between the two denominators is a power of an odd number, power that increased by one has to be even–even. Q. E. D.

Corollary 1. *As these powers of odd numbers are such disposed that, increased by one, become divisible by 4, they will be powers of odd dimension, coming from numbers of the form* $4m - 1$ *which are not powers themselves.*

---

[3]The original omits the dots. P.V. *et altri.*

[4]Euler seems to explain here what is a number *divisible by four.* To our modern eyes it clarifies the meaning of *even–even* numbers. P.V. *et altri.*

Corollary 2. *Consequently, if we take all numbers of the form $4m-1$ which are not powers and among them we choose all the powers with an odd exponent, these powers either increased or decreased by one will be all the denominators of the fractions of the series found.*

Corollary 3. *If we add the fractions by twos, we have*

$$\frac{\pi}{4} = l2 + \frac{2 \cdot 27}{26 \cdot 28} + \frac{2 \cdot 243}{242 \cdot 244} + \frac{2 \cdot 343}{342 \cdot 344} + \cdots$$

series which will be formed by adding up all the fractions that come from the formula

$$\frac{2(4m-1)^{2n+1}}{(4m-1)^{4n+2}-1}$$

replacing m and n by all the consecutive integers excepting those values of m that make $4m-1$ a power.

Theorem 6. *The series*

$$\frac{1}{15} + \frac{1}{63} + \frac{1}{80} + \frac{1}{255} + \frac{1}{624} + \cdots,$$

whose denominators increased by one are all the squares that are at the same time higher powers, continued infinitely has sum $\frac{7}{4} - \frac{\pi^2}{6}$, where π denotes the perimeter of the circle of diameter one.

Proof. I also received this Theorem from Cel. Goldbach[5], though without proof, but insisting with the same methods I found the following proof.

Some years ago[6] I found that the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

summed $\pi^2/6$, and this very series I saw in this way

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots.$$

Now, as

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$$

and also

$$\frac{1}{8} = \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \cdots$$

and in the same way

$$\frac{1}{24} = \frac{1}{25} + \frac{1}{625} + \cdots \qquad \text{and} \qquad \frac{1}{35} = \frac{1}{36} + \cdots,$$

if, instead of these geometrical series we replace their sums, we get

$$\frac{\pi^2}{6} = 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \frac{1}{99} + \cdots,$$

---

[5] See note 1 on page 1. C.B.

[6] See Comment # 41 of this volume, p. 85. C.B.

[It is "De summis serierum reciprocarum", comment # 41 from Enestroemiani's Index. *Commentarii academiae scientarum Petropolitanae* **7** (1734/5), **1740**, 123–134. On the page quoted one can see

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{p^2}{6}.$$

Here Euler used p for π. P.V. *et altri.*]

series whose denominators, increased by one are all the square numbers except for those which at the same time are powers of other orders. But if we consider only the squares decreased by one, we have

$$\frac{3}{4} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \frac{1}{63} + \frac{1}{80} + \cdots$$

and subtracting from this series the series above, we have

$$\frac{7}{4} - \frac{\pi^2}{6} = \frac{1}{15} + \frac{1}{63} + \frac{1}{80} + \frac{1}{255} + \cdots$$

whose denominators, increased by one are all the square numbers that, at the same time, are powers of other orders. Q. E. D.

These six Theorems constitute the first part of these remarks where the series are obviously considered generated by addition or subtraction of terms. The following Theorems will deal with series whose terms will multiply each other and will not be less admirable than the former as in them, the law of progression is also as irregular. There is, though, a fundamental difference as in the former Theorems the progression of terms followed the series of the powers, in itself quite irregular. In these Theorems the terms will progress according the prime numbers whose progression is not less abstruse.

Theorem 7. *If we take to the infinity the continuation of these fractions*

$$\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \cdots},$$

where the numerators are all the prime numbers and the denominators are the numerators less one unit, the result is the same as the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

which is certainly infinity.

Proof. If we have

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

then we will have

$$\frac{1}{2}x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots,$$

which subtracted from the first will leave us with

$$\frac{1}{2}x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots,$$

series where there are no even denominators. From this one, we subtract again this series

$$\frac{1}{2} \cdot \frac{1}{3}x = \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \cdots;$$

and we will have

$$\frac{1}{2} \cdot \frac{2}{3}x = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots,$$

where among the denominators we cannot find any divisible either by 2 or by 3. In order to remove the numbers divisible by 5, we subtract the following series

$$\frac{1 \cdot 2}{2 \cdot 3} \cdot \frac{1}{5}x = \frac{1}{5} + \frac{1}{25} + \frac{1}{35} + \cdots$$

and we will have

$$\frac{1 \cdot 2 \cdot 4}{2 \cdot 3 \cdot 5}x = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots.$$

And proceeding in the same way, subtracting all the terms divisible now by 7, now by 11, and now by all the prime numbers we finally have

$$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \cdot 22 \cdots}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdots} x = 1.$$

As

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots,$$

we have

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \cdot 22 \cdots},$$

expression whose numerators constitute the sequence of prime numbers and the denominators are the same less one unit. Q. E. D.

Corollary 1. *Thus, the value of the expression*

$$\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdots}$$

is infinity, and if we denote the absolute infinity as ∞, the value of this expression is $l\infty$, which is the minimum among all the powers of the infinity.

Corollary 2. *As the expression*

$$\frac{4 \cdot 9 \cdot 16 \cdot 25 \cdot 36 \cdot 49 \cdots}{3 \cdot 8 \cdot 15 \cdot 24 \cdot 35 \cdot 48 \cdots}$$

has a finite value, which is 2, it follows that the prime numbers are infinitely many times more numerous than the squares in the sequence of all numbers.

Corollary 3. *Also from here it is also true that the primer numbers are infinitely many less numerous than the whole numbers as the value of the expression*

$$\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots}$$

is the absolute infinity, and the similar value from the prime numbers is the logarithm of this value.

Theorem 8. *The expression formed from the sequence of prime numbers*

$$\frac{2^n \quad \cdot \quad 3^n \quad \cdot \quad 5^n \quad \cdot \quad 7^n \quad \cdot \quad 11^n \quad \cdots}{(2^n-1)\,(3^n-1)\,(5^n-1)\,(7^n-1)\,(11^n-1)\,\cdots}$$

has the same value as the sum of the series

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \cdots$$

Proof. Let

$$x = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \cdots;$$

we will have

$$\frac{1}{2^n} x = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \cdots,$$

from where

$$\frac{2^n-1}{2^n} x = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \cdots.$$

We also have

$$\frac{2^n-1}{2^n} \cdot \frac{1}{3^n} x = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \cdots,$$

from where

$$\frac{(2^n-1)(3^n-1)}{2^n \cdot 3^n} x = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \cdots.$$

Consequently, carrying out the same procedure with each prime number one by one, all the terms of the series will disappear except for the first, and we will have

$$1 = \frac{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\cdots}{2^n\cdot 3^n\cdot 5^n\cdot 7^n\cdot 11^n\cdots}x,$$

and replacing x by its series,

$$\frac{2^n\cdot 3^n\cdot 5^n\cdot 7^n\cdot 11^n\cdots}{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\cdots} = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\cdots.$$

Q. E. D.

Corollary 1. *If we make $n = 2$, as*

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = \frac{\pi^2}{6},$$

where π denotes the perimeter of the circle whose diameter is one, we will have

$$\frac{4\cdot 9\cdot 25\cdot 49\cdot 121\cdot 169\cdots}{3\cdot 8\cdot 24\cdot 48\cdot 120\cdot 168\cdots} = \frac{\pi^2}{6}$$

or

$$\frac{\pi^2}{6} = \frac{2\cdot 2\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 11\cdot 11\cdots}{1\cdot 3\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 10\cdot 12\cdots}.$$

Corollary 2. *As before if we make $n = 4$, as*

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots = \frac{\pi^4}{90},$$

we will have

$$\frac{\pi^4}{90} = \frac{4\cdot 4\cdot 9\cdot 9\cdot 25\cdot 25\cdot 49\cdot 49\cdot 121\cdot 121\cdots}{3\cdot 5\cdot 8\cdot 10\cdot 24\cdot 26\cdot 48\cdot 50\cdot 120\cdot 122\cdots}.$$

This expression divided by the former will give us

$$\frac{\pi^2}{15} = \frac{4\cdot 9\cdot 25\cdot 49\cdot 121\cdot 169\cdots}{5\cdot 10\cdot 26\cdot 50\cdot 122\cdot 170\cdots}.$$

Theorem 9. *If the squares of all the prime numbers are separated in two different parts with one unit of difference between them, and we take the odd parts as numerators and the even parts as denominators of the series formed by these factors, the value of this expression will be*

$$\frac{5\cdot 13\cdot 25\cdot 61\cdot 85\cdot 145\cdots}{4\cdot 12\cdot 24\cdot 60\cdot 84\cdot 144\cdots} = \frac{3}{2}.$$

Proof. By Corollary 1 of the previous Theorem we have

$$\frac{\pi^2}{6} = \frac{4\cdot 9\cdot 25\cdot 49\cdot 121\cdot 169\cdot 289\cdots}{3\cdot 8\cdot 24\cdot 48\cdot 120\cdot 168\cdot 288\cdots}.$$

But in Corollary 2 we deduced the following equation

$$\frac{\pi^2}{15} = \frac{4\cdot 9\cdot 25\cdot 49\cdot 121\cdot 169\cdot 289\cdots}{5\cdot 10\cdot 26\cdot 50\cdot 122\cdot 170\cdot 290\cdots}.$$

If we divide this expression by the other one we will have

$$\frac{5}{2} = \frac{5\cdot 10\cdot 26\cdot 50\cdot 122\cdot 170\cdot 290\cdots}{3\cdot 8\cdot 24\cdot 48\cdot 120\cdot 168\cdot 288\cdots},$$

expression whose numerators are the squares of the prime numbers increased by one unit and the denominators decreased by one unit. Dividing both sides by 5/3 and simplifying by 2 each fraction, one by one, we have

$$\frac{3}{2} = \frac{5\cdot 13\cdot 25\cdot 61\cdot 85\cdot 145\cdots}{4\cdot 12\cdot 24\cdot 60\cdot 84\cdot 144\cdots},$$

where the numerators are one unit greater than their corresponding denominators and each numerator added with its denominator is the square of an odd prime number, as in the simplification process the square of the even prime number 2 has disappeared. Q. E. D.

Theorem 10. *If π denotes the perimeter of the circle whose diameter is one, we have*
$$\frac{\pi^3}{32} = \frac{80 \cdot 224 \cdot 440 \cdot 624 \cdot 728 \cdots}{81 \cdot 225 \cdot 441 \cdot 625 \cdot 729 \cdots},$$
expression whose denominators are the squares of the odd numbers not prime and whose denominators are these same numbers less one.

Proof. We owe the following expression for π to Wallis[7]
$$\frac{\pi}{4} = \frac{8 \cdot 24 \cdot 48 \cdot 80 \cdot 120 \cdot 168 \cdots}{9 \cdot 25 \cdot 49 \cdot 81 \cdot 121 \cdot 169 \cdots},$$
whose fractions are formed only by all the squares of the odd numbers. By Corollary 1 of Theorem 8 we have
$$\frac{\pi^2}{6} = \frac{4 \cdot 9 \cdot 25 \cdot 49 \cdot 121 \cdot 169 \cdots}{3 \cdot 8 \cdot 24 \cdot 48 \cdot 120 \cdot 168 \cdots}$$
or
$$\frac{\pi^2}{8} = \frac{9 \cdot 25 \cdot 49 \cdot 121 \cdot 169 \cdot 289 \cdots}{8 \cdot 24 \cdot 48 \cdot 120 \cdot 168 \cdot 288 \cdots},$$
whose fractions are formed only by the squares of the odd numbers. Now, if we multiply these two expressions we have
$$\frac{\pi^3}{32} = \frac{80 \cdot 224 \cdot 440 \cdot 624 \cdot 728 \cdots}{81 \cdot 225 \cdot 441 \cdot 625 \cdot 729 \cdots},$$
whose fractions happen to be the squares of the odd numbers not prime. Q. E. D.

Theorem 11. *If we take π as the perimeter of the circle whose diameter is one, we have*
$$\frac{\pi}{4} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdots}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 20 \cdot 24 \cdots},$$
expression whose numerators are the sequence of the prime numbers and whose denominators are even–even numbers one unit more or less than the corresponding numerators.

Proof. As
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \cdots$$
we will have
$$\frac{1}{3} \cdot \frac{\pi}{4} = \frac{1}{3} - \frac{1}{9} + \frac{1}{15} - \frac{1}{21} + \cdots,$$
and adding up both series,
$$\frac{4}{3} \cdot \frac{\pi}{4} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \cdots.$$
We also have
$$\frac{1}{5} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = \frac{1}{5} + \frac{1}{25} - \frac{1}{35} - \frac{1}{55} + \cdots,$$
that subtracted from the former leads to
$$\frac{4}{5} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = 1 - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \cdots,$$
series where there are no denominators divisible either by 3 or by 5. In a similar way we remove all those divisible by 7,
$$\frac{1}{7} \cdot \frac{4 \cdot 4}{5 \cdot 3} \cdot \frac{\pi}{4} = \frac{1}{7} - \frac{1}{49} - \frac{1}{77} + \cdots;$$

[7]See note 1 on page 3. C.B.

which once subtracted gives

$$\frac{8\cdot 4\cdot 4}{7\cdot 5\cdot 3}\cdot\frac{\pi}{4}=1-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\cdots.$$

We notice that the denominators which are divisible by a prime number of the form $4n-1$ are removed by addition, from which this new factor gets added, $\dfrac{4n}{4n-1}$, while the denominators divisible by a prime number of the form $4n+1$ are removed by subtraction from which this new factor gets added, $\dfrac{4n}{4n+1}$. Thus the denominators of these new factors successively added will be prime numbers while the numerators will be even–even numbers, one unit more or less than the denominators. Consequently if all the terms of the series considered at the beginning are subtracted in this way, we will finally have

$$\frac{\cdots 24\cdot 20\cdot 16\cdot 12\cdot 12\cdot 8\cdot 4\cdot 4}{\cdots 23\cdot 19\cdot 17\cdot 13\cdot 11\cdot 7\cdot 5\cdot 3}\cdot\frac{\pi}{4}=1.$$

From this we will get

$$\frac{\pi}{4}=\frac{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdots}{4\cdot 4\cdot 8\cdot 12\cdot 12\cdot 16\cdot 20\cdot 24\cdots}.$$

Q. E. D.

Theorem 12. *If all the odd prime numbers are separated in two parts, one of the parts one unit greater than the other and we take the even parts as numerators and the odd ones as denominators, we will get the continued product*

$$\frac{2\cdot 2\cdot 4\cdot 6\cdot 6\cdot 8\cdot 10\cdot 12\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 7\cdot 9\cdot 9\cdot 11\cdots}=2.$$

Proof. As by the previous Theorem

$$\frac{\pi}{4}=\frac{3\cdot 5\cdot 7\cdots}{4\cdot 4\cdot 8\cdots},$$

we will have

$$\frac{16}{\pi^2}=\frac{4\cdot 4\cdot 4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12\cdot 12\cdot 12\cdot 16\cdot 16\cdots}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 11\cdot 11\cdot 13\cdot 13\cdot 17\cdot 17\cdots}.$$

But, by Corollary 1 of Theorem 8, if multiplied by 3/4 we have

$$\frac{\pi^2}{8}=\frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 11\cdot 11\cdot 13\cdot 13\cdots}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 10\cdot 12\cdot 12\cdot 14\cdots},$$

and both expressions are formed by odd prime numbers. If we multiply them reciprocally, the denominator of the first one will cancel the numerator of the second one and, moreover, the central part of the terms of both the numerator of the former and the denominator of the latter will vanish. It will remain,

$$2=\frac{4\cdot 4\cdot 8\cdot 12\cdot 12\cdot 16\cdot 20\cdot 24\cdots}{2\cdot 6\cdot 6\cdot 10\cdot 14\cdot 18\cdot 18\cdot 22\cdots},$$

where the numerators are eve–even numbers and the denominators odd–even numbers, both one unit greater or less than the odd prime numbers.

If we now simplify by two the fractions one by one, the numerators will be even numbers and the denominators odd; matching couples will differ by one and added together will be prime numbers. Consequently we will have

$$2=\frac{2\cdot 2\cdot 4\cdot 6\cdot 6\cdot 8\cdot 10\cdot 12\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 7\cdot 9\cdot 9\cdot 11\cdots}.$$

Q. E. D.

Theorem 13. *If all the odd prime numbers are separated into two parts, one a unit greater than the other and we use the even parts as numerators and the odd parts as denominators we will have*

$$\frac{\pi}{4} = \frac{4 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \cdot 18 \cdot 20 \cdot 22 \cdot 24 \cdots}{5 \cdot 7 \cdot 11 \cdot 13 \cdot 13 \cdot 17 \cdot 17 \cdot 19 \cdot 23 \cdot 25 \cdots}.$$

Proof. By WALLIS' quadrature of the circle we have

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdots},$$

expression in which if we add all numerators one by one to their corresponding denominators all odd numbers result. Now, as from the previous Theorem a similar expression formed only by odd prime numbers has the value 2,

$$2 = \frac{2 \cdot 2 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdots},$$

dividing the former expression by this one we have

$$\frac{\pi}{4} = \frac{4 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \cdot 18 \cdot 20 \cdot 22 \cdot 24 \cdots}{5 \cdot 7 \cdot 11 \cdot 13 \cdot 13 \cdot 17 \cdot 17 \cdot 19 \cdot 23 \cdot 25 \cdots},$$

which, similarly, is formed by odd numbers not prime. That is to say, the numerators are even numbers and the denominators are odd numbers one unit from the numerators and such that, added together one by one, we get all the odd numbers not prime. Q. E. D.

Theorem 14. *If π denotes the perimeter of the circle whose diameter is one, I say we will have*

$$\frac{\pi}{2} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdots}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 18 \cdot 22 \cdot 30 \cdot 30 \cdots},$$

expression whose numerators constitute the sequence of all odd prime numbers and the denominators are odd–even numbers one unit greater or less than the corresponding numerators.

Proof. By Corollary 1 of Theorem 8, if multiplied by 3/4 we have

$$\frac{\pi^2}{8} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdot 13 \cdot 13 \cdots}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 12 \cdot 14 \cdots},$$

where the numerators are twice the odd prime numbers and the denominators are either even–even numbers or odd–even numbers one unit greater or lesser than the same prime numbers. Besides, by Theorem 11 we had

$$\frac{\pi}{4} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdots}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 20 \cdot 24 \cdots},$$

expression whose numerators are the odd prime numbers once and whose denominators are even–even numbers one unit apart from the prime numbers. Thus this expression is contained in the former and if we divide the former by this one we have

$$\frac{\pi}{2} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdots}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 18 \cdots},$$

where the odd prime numbers are the numerators and the denominators are odd–even numbers one unit greater or less than the numerators. Q. E. D.

Theorem 15. *Being π the perimeter of the circle whose diameter is one, we have*

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} - \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23}$$
$$+ \frac{1}{25} + \frac{1}{27} - \frac{1}{29} + \frac{1}{31} + \frac{1}{33} - \frac{1}{35} - \frac{1}{37} - \cdots,$$

series whose denominators are all the odd numbers and whose signs obey to the following law: consider those prime numbers of the form $4n-1$ with the sign $+$ and those prime numbers of the form $4n+1$ with the sign $-$. Composite numbers bear the sign that correspond to them according the rules of product applied to their prime number factorization.

Proof. In the same way that, with the operations we use, this series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\cdots,$$

becomes the expression

$$\frac{3\cdot 5\cdot 7\cdot 11\cdots}{4\cdot 4\cdot 8\cdot 12\cdots},$$

we can again devise another method to transform the expression

$$\frac{3\cdot 5\cdot 7\cdot 11\cdot 13\cdots}{4\cdot 4\cdot 8\cdot 12\cdot 12\cdots}$$

into the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\cdots.$$

Applying this method to the expression found in the previous Theorem,

$$\frac{\pi}{2}=\frac{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdots}{2\cdot 6\cdot 6\cdot 10\cdot 14\cdot 18\cdots},$$

this expression will become the series we contend

$$1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\cdots,$$

whose sum is $\pi/2$. The same result can be reached *a posteriori* writing

$$x=1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}-\frac{1}{17}+\cdots,$$

from where

$$\frac{1}{3}x=\frac{1}{3}+\frac{1}{9}-\frac{1}{15}+\frac{1}{21}+\frac{1}{27}+\frac{1}{33}-\cdots.$$

By subtraction we have

$$\frac{2}{3}x=1-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\cdots.$$

Now, in the same way,

$$\frac{1}{5}\cdot\frac{2}{3}x=\frac{1}{5}-\frac{1}{25}+\frac{1}{35}+\frac{1}{55}-\cdots,$$

which added to the former,

$$\frac{6\cdot 2}{5\cdot 3}x=1+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\cdots.$$

Removing in the same way all the terms except for the first 1, we have

$$x=\frac{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdots}{2\cdot 6\cdot 6\cdot 10\cdot 14\cdot 18\cdot 18\cdots}=\frac{\pi}{2}.$$

And from here, at the same time we will obtain the law of signs described in the statement of the Theorem. Q. E. D.

Corollary 1. *The sum of the proposed series*

$$1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\cdots$$

is twice the sum of this series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots.$$

As the fractions are the same in both series, the only reason for one being double than the other must be found in the signs.

Theorem 16. *Let π be the perimeter of the circle whose diameter is one. We have*

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{6} + \frac{1}{10} - \frac{1}{14} - \frac{1}{16} - \frac{1}{18} + \frac{1}{18} + \frac{1}{20} + \cdots .$$

The denominators of the positive fractions are one unit less than the odd numbers which are not powers and the denominators of the negative fractions are one unit greater. The sign of each fraction is in accordance with the sign of the odd number one unit greater or less not a power obtained through the law of the previous Theorem.

Proof. This very series is obtained from the conversion of the previous following the method of Theorems 1, 2 and 3 according to which geometrical progressions are added or subtracted from an expression until only the first term remains. Q. E. D.

Theorem 17. *If odd primes of the form $4n-1$ are given a $+$ sign, and the rest of the form $4n+1$ are given the sign $-$ and to the composite numbers are given the signs that are obtained from their prime number factorization obeying the rule of product, we will have*

$$\frac{3\pi}{8} = 1 + \frac{1}{9} - \frac{1}{15} + \frac{1}{21} + \frac{1}{25} + \frac{1}{33} - \frac{1}{35} - \frac{1}{39} + \frac{1}{49} - \frac{1}{51} - \cdots ,$$

whose denominators are constituted by 2, then 4, then 6, etc. prime numbers.

Proof. By Theorem 15 we have

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} - \frac{1}{17} + \frac{1}{19} + \cdots ,$$

where the denominators are all the odd numbers and the signs follow the law we have stated above; moreover we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \cdots ;$$

the terms of these series have the signs that their denominators command, whether they are made of 2 or 4 or 6 prime numbers, etc. Thus adding them up only those terms will remain and then, dividing by 2, we have

$$\frac{3\pi}{8} = 1 + \frac{1}{9} - \frac{1}{15} + \frac{1}{21} + \frac{1}{25} + \frac{1}{33} - \cdots ,$$

which is the very series proposed. And by the rule of signs it follows that those fractions whose denominators is of the form $4n+1$, have a $+$ sign, and the rest, have sign $-$. Q. E. D.

Corollary 1. *If from the series of the Theorem we subtract this one,*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \cdots ,$$

we will have the series

$$\frac{\pi}{8} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{17} + \frac{1}{19} + \cdots ,$$

whose denominators are either the prime numbers themselves or are constituted by 3 or 5, etc.; and those terms of the form $4n-1$ have a $+$ sign and the rest of the form $4n+1$ have sign $-$.

Theorem 18. *If we assign a $-$ sign to all the prime numbers and composite numbers are assigned the sign that correspond to them according to the rule of signs in the product and with all the numbers we form the series*

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\frac{1}{12}-\cdots$$

will have, once infinitely continued, sum 0.

Proof. Let x be the sum of the series, that is

$$x=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots;$$

we will have, operating as we did in previous Theorems[8],

$$\frac{3}{2}x=1-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\cdots;$$

and, similarly,

$$\frac{3}{2}\cdot\frac{4}{3}x=1-\frac{1}{5}-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}-\cdots.$$

Finally, repeating the same operations infinitely many times,

$$\frac{3\cdot 4\cdot 6\cdot 8\cdot 12\cdot 14\cdots}{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdots}x=1.$$

Now, by Theorem 7,

$$\frac{2\cdot 3\cdot 5\cdot 7\cdot 11\cdots}{1\cdot 2\cdot 4\cdot 6\cdot 10\cdots}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=l\infty,$$

and it is easily seen that also the coefficient of our x is infinitely great. Thus, in order to be able to equal 1, x must be 0 and so

$$0=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots,$$

whose denominators which either prime numbers themselves or which are formed by 3, 5, 7, etc. prime factors have a $-$ sign and the rest a $+$ sign. Q. E. D.

Corollary 1. *It is thus obvious the way in which in the harmonic progression the signs have to be distributed in order to have 0 as the sum of the whole series.*

Corollary 2. *If we have found $x=0$, also $\frac{3}{2}x=0$ and, consequently, we will have*

$$0=1-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+\frac{1}{15}-\cdots,$$

where only the odd numbers occur and the law of signs is as described above.

Theorem 19. *The sum of the reciprocals of the prime numbers,*

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\cdots$$

is infinitely great but is infinitely times less than the sum of the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots.$$

And the sum of the former is as the logarithm of the sum of the latter.

[8]See, for example, page 8. C.B.

Proof. If we write

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots=A$$

and

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\cdots=B$$

and

$$\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{11^3}+\cdots=C$$

and thus successively we denote all the powers by their corresponding letters, we will have, calling e that number whose hyperbolic logarithm is 1,

$$e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\cdots}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots.$$

As

$$A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\cdots=l\frac{2}{1}+l\frac{3}{2}+l\frac{5}{4}+l\frac{7}{6}+\cdots,$$

and consequently

$$e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\cdots}=\frac{2\cdot3\cdot5\cdot7\cdots}{1\cdot2\cdot4\cdot6\cdots}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$$

by Theorem 7. But not only B, C, D, etc. will have finite values, but also

$$\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\cdots$$

will have a finite value. Thus in order to have $e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\cdots}$ equal to

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\infty$$

it is necessary that A be an infinitely great quantity and thus with respect to it, the following terms

$$\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\cdots$$

will vanish and we will have

$$e^A=e^{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots.$$

Hence, we will have

$$\begin{aligned}\frac{1}{2}+&\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots\\ &=l\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\cdots\right)\end{aligned}$$

and the sum of the former series[9] will be infinitely times less than this one whose sum is $l\infty$ and finally,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots=l.l\infty.$$

Q. E. D.

[9]There is a small mistake in the last series. It should be

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots.$$

P.V. *et altri.*

Translated from Latin by Pelegrí Viader and Lluís Bibiloni and Pelegrí Viader (Jr.)
Pelegrí Viader (Retired).

Lluís Bibiloni. lluis.bibiloni@uab.es
Facultat de Ciències de l'Educació
Univ. Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain.

Pelegrí Viader (Jr.). pelegri.viader@econ.upf.es
Departament d'Economia i Empresa
Universitat Pompeu Fabra
c/ Ramon Trias Fargas 25–27
08005 Barcelona, Spain.