Excerpts on the Euler-Maclaurin summation formula, from
Institutiones Calculi Differentialis by Leonhard Euler.
Translated by David Pengelley

Leonhard Euler's book *Institutiones Calculi Differentialis* (Foundations of Differential Calculus) was published in two parts in 1755 [2, series I, vol. 10], and translated into German in 1790 [3]. There is an English translation of part 1 [4], but not part 2, of the *Institutiones.*

Euler was entranced by infinite series, and a wizard at working with them. A lot of the book is actually devoted to the relationship between differential calculus and infinite series, and in this respect it differs considerably from today's calculus books. In chapters 5 and 6 of part 2 Euler presents his way of finding sums of series, both finite and infinite, via his discovery of the Euler-Maclaurin summation formula.

Here we present translated excerpts from these two chapters of part 2, featuring aspects of Euler's development and applications of his summation formula. The English translation has been made primarily from the German translation of 1790 [3], with some assistance from Daniel Otero in comparing the resulting English with the Latin original. The excerpts include those in a forthcoming book of annotated original sources, within a chapter on *The Bridge Between the Continuous and Discrete.* The book chapter follows this theme via sources by Archimedes, Fermat, Pascal, Bernoulli and Euler. For more information see `http://math.nmsu.edu/~history`. In the spirit of providing pure uninterpreted translation, we have here removed all our annotation and commentary, which along with extensive exercises can be found in the book chapter. The version with annotation and exercises may be provided at this site at a later time. The only commentary that remains here summarizes some of those portions of Chapters 5 and 6 that were not translated.

In excerpts from chapter 5 we see Euler derive his summation formula, analyze the nature of its Bernoulli numbers in connection with trigonometric functions, find the precise sums of infinite series of reciprocal even powers, and prove Bernoulli's sums of powers formulas. From chapter 6 we see three diverse applications of the summation formula, each revealing a fundamentally different way of using it. We first see Euler approximate large partial sums of the slowly diverging harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$, which involves approximating the now famous "Euler constant". Then we see how in the early 1730's Euler approximated the infinite sum of reciprocal squares to great precision without knowledge of the infinite sum itself. Finally Euler goes on to use the summation formula to study sums of logarithms, from which he obtains incredibly impressive formulas and approximations for large factorials (Stirling's series), and thence for binomial coefficients, using Wallis's formula for π to determine the unknown constant in his summation formula.

Leonhard Euler, from *Foundations of Differential Calculus*

Part Two, Chapter 5
On Finding Sums of Series from the General Term

103. Suppose y is the general term of a series, belonging to the index x, and thus y is any function of x. Further, suppose Sy is the summative term of this series, expressing the aggregate of all terms from the first or another fixed term up to y, inclusive. The sums of the series are calculated from the first term, so that if $x = 1$, y is the first term, and likewise Sy yields this first term; alternatively, if $x = 0$, the summative term Sy vanishes, because no terms are being summed. With these stipulations, the summative term Sy is a function of x that vanishes if one sets $x = 0$.

[...]

105. Consider a series whose general term, belonging to the index x, is y, and whose preceding term, with index $x-1$, is v; because v arises from y, when x is replaced by $x-1$, one has

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.}$$

If y is the general term of the series

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & x-1 & x \\ a+ & b+ & c+ & d+ & \cdots + & v & +y \end{array}$$

and if the term belonging to the index 0 is A, then v, as a function of x, is the general term of the series

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & \cdots & x \\ A+ & a+ & b+ & c+ & d+ & \cdots + & v \end{array},$$

so if Sv denotes the sum of this series, then $Sv = Sy - y + A$. If one sets $x = 0$, then $Sy = 0$ and $y = A$, so Sv vanishes.

106. Because

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \text{etc.},$$

one has, from the preceding,

$$Sv = Sy - S\frac{dy}{dx} + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \text{ etc.},$$

and, because $Sv = Sy - y + A$,

$$y - A = S\frac{dy}{dx} - S\frac{ddy}{2dx^2} + S\frac{d^3y}{6dx^3} - S\frac{d^4y}{24dx^4} + \text{etc.},$$

or equivalently

$$S\frac{dy}{dx} = y - A + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \text{etc.}$$

Thus if one knows the sums of the series, whose general terms are $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, etc., one can obtain the summative term of the series whose general term is $\frac{dy}{dx}$. The constant A must then be such that the summative term $S\frac{dy}{dx}$ disappears when $x = 0$, and this condition makes it easier to determine, than saying that it is the term belonging to the index 0 in the series whose general term is y.

In §107/108 Euler illustrates the practical application of this equation by choosing to use the power function $y = x^{n+1}/(n+1)$. This has the advantage that the derivatives in the equation are just lower power functions, so that the sums are all sums of powers, and then vanish after some point in the equation, so he obtains a finite expression for Sx^n (i.e., for $\sum_{i=1}^{x} i^n$). He applies this inductively from $n = 0$ upwards to calculate the closed formulae for sums of powers of the natural numbers explicitly up through the sum of fourth powers.

[...]

109. Since from the above one has

$$S\frac{dy}{dx} = y\ [-A] + \frac{1}{2}S\frac{ddy}{dx^2} - \frac{1}{6}S\frac{d^3y}{dx^3} + \frac{1}{24}S\frac{d^4y}{dx^4} - \frac{1}{120}S\frac{d^5y}{dx^5} + \text{etc.},$$

if one sets $\frac{dy}{dx} = z$, then $\frac{ddy}{dx^2} = \frac{dz}{dx}$, $\frac{d^3y}{dx^3} = \frac{ddz}{dx^2}$, etc. And because $dy = zdx$, y will be a quantity whose differential is zdx, and this one writes as $y = \int zdx$. Now the determination of the quantity y from z according to this formula assumes the integral calculus; but we can nevertheless make use of this expression $\int zdx$, if for z we use no function other than that whose differential is zdx from above. Thus substituting these values yields

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \text{etc.},$$

adding to it a constant value such that when $x = 0$, the sum Sz also vanishes.

110. But if in the expressions above one substitutes the letter z in place of y, or if one differentiates the preceding equation, which yields the same, one obtains

$$S\frac{dz}{dx} = z + \frac{1}{2}S\frac{ddz}{dx^2} - \frac{1}{6}S\frac{d^3z}{dx^3} + \frac{1}{24}S\frac{d^4z}{dx^4} - \text{etc.};$$

but using $\frac{dz}{dx}$ in place of y one obtains

$$S\frac{ddz}{dx^2} = \frac{dz}{dx} + \frac{1}{2}S\frac{d^3z}{dx^3} - \frac{1}{6}S\frac{d^4z}{dx^4} + \frac{1}{24}S\frac{d^5z}{dx^5} - \text{etc.}$$

Similarly replacing y successively by the values $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc., produces

$$S\frac{d^3z}{dx^3} = \frac{ddz}{dx^2} + \frac{1}{2}S\frac{d^4z}{dx^4} - \frac{1}{6}S\frac{d^5z}{dx^5} + \frac{1}{24}S\frac{d^6z}{dx^6} - \text{etc.},$$
$$S\frac{d^4z}{dx^4} = \frac{d^3z}{dx^3} + \frac{1}{2}S\frac{d^5z}{dx^5} - \frac{1}{6}S\frac{d^6z}{dx^6} + \frac{1}{24}S\frac{d^7z}{dx^7} - \text{etc.},$$

and so forth indefinitely.

111. Now when these values for $S\frac{dz}{dx}$, $S\frac{ddz}{dx^2}$, $S\frac{d^3z}{dx^3}$ are successively substituted in the expression

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \text{etc.},$$

one finds an expression for Sz, composed of the terms $\int zdx$, z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc., whose coefficients are easily obtained as follows. One sets

$$Sz = \int zdx + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \text{etc.},$$

and substitutes for these terms the values they have from the previous series, yielding

$$\begin{aligned}
\textstyle\int zdx &= Sz - \tfrac{1}{2}S\tfrac{dz}{dx} + \tfrac{1}{6}S\tfrac{ddz}{dx^2} - \tfrac{1}{24}S\tfrac{d^3z}{dx^3} + \tfrac{1}{120}S\tfrac{d^4z}{dx^4} - \text{etc.}\\
\alpha z &= \quad\; + \alpha S\tfrac{dz}{dx} - \tfrac{\alpha}{2}S\tfrac{ddz}{dx^2} + \tfrac{\alpha}{6}S\tfrac{d^3z}{dx^3} - \tfrac{\alpha}{24}S\tfrac{d^4z}{dz^4} + \text{etc.}\\
\tfrac{\beta dz}{dx} &= \qquad\qquad\quad \beta S\tfrac{ddz}{dx^2} - \tfrac{\beta}{2}S\tfrac{d^3z}{dx^3} + \tfrac{\beta}{6}S\tfrac{d^4z}{dx^4} - \text{etc.}\\
\tfrac{\gamma ddz}{dx^2} &= \qquad\qquad\qquad\qquad \gamma S\tfrac{d^3z}{dx^3} - \tfrac{\gamma}{2}S\tfrac{d^4z}{dx^4} + \text{etc.}\\
\tfrac{\delta d^3z}{dx^3} &= \qquad\qquad\qquad\qquad\qquad\quad \delta\, S\tfrac{d^4z}{dx^4} - \text{etc.}\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

Since these values, added together, must produce Sz, the coefficients α, β, γ, δ etc. are defined by the sequence of equations

$$\alpha - \frac{1}{2} = 0, \quad \beta - \frac{\alpha}{2} + \frac{1}{6} = 0, \quad \gamma - \frac{\beta}{2} + \frac{\alpha}{6} - \frac{1}{24} = 0,$$
$$\delta - \frac{\gamma}{2} + \frac{\beta}{6} - \frac{\alpha}{24} + \frac{1}{120} = 0, \quad \varepsilon - \frac{\delta}{2} + \frac{\gamma}{6} - \frac{\beta}{24} + \frac{\alpha}{120} - \frac{1}{720} = 0,$$
$$\zeta - \frac{\varepsilon}{2} + \frac{\delta}{6} - \frac{\gamma}{24} + \frac{\beta}{120} - \frac{\alpha}{720} + \frac{1}{5040} = 0 \quad \text{etc.}$$

112. So from these equations the successive values of all the letters α, β, γ, δ etc. are defined; they are

$$\alpha = \frac{1}{2}, \quad \beta = \frac{\alpha}{2} - \frac{1}{6} = \frac{1}{12}, \quad \gamma = \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24} = 0,$$
$$\delta = \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120} = -\frac{1}{720}, \quad \varepsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720} = 0 \quad \text{etc.},$$

and if one continues in this fashion one finds that alternating terms vanish. The third, fifth, seventh letters, and so on, in fact all odd terms except the first, are zero, so that this series appears to contradict the law of continuity by which the terms proceed. A rigorous proof is especially needed that all odd terms except the first vanish.

113. Because the letters are determined from the preceding by a constant law, they form a recurrent series. In order to develop this, consider the series

$$1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.},$$

and set its value $=V$, so it is clear that this recurrent series arises from the development of the fraction

$$V=\frac{1}{1-\frac{1}{2}u+\frac{1}{6}u^2-\frac{1}{24}u^3+\frac{1}{120}u^4-\text{etc.}}$$

And when this fraction is resolved in a different way in an infinite series according to the powers of u, then necessarily the same series

$$V=1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\text{etc.}$$

will always result. In this fashion a different rule for determining the letters α, β, γ, δ etc. results.

114. Because one has

$$e^{-u}=1-u+\frac{1}{2}u^2-\frac{1}{6}u^3+\frac{1}{24}u^4-\frac{1}{120}u^5+\text{etc.},$$

where e denotes the number whose hyperbolic logarithm is one, then

$$\frac{1-e^{-u}}{u}=1-\frac{1}{2}u+\frac{1}{6}u^2-\frac{1}{24}u^3+\frac{1}{120}u^4-\text{etc.},$$

and thus

$$V=\frac{u}{1-e^{-u}}.$$

Now one removes from this series the second term $\alpha u=\frac{1}{2}u$, so that

$$V-\frac{1}{2}u=1+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.};$$

whence

$$V-\frac{1}{2}u=\frac{\frac{1}{2}u\,(1+e^{-u})}{1-e^{-u}}.$$

Multiplying numerator and denominator by $e^{\frac{1}{2}u}$ yields

$$V-\frac{1}{2}u=\frac{u\left(e^{\frac{1}{2}u}+e^{-\frac{1}{2}u}\right)}{2\left(e^{\frac{1}{2}u}-e^{-\frac{1}{2}u}\right)},$$

and converting the quantities $e^{\frac{1}{2}u}$ and $e^{-\frac{1}{2}u}$ into series gives

$$V-\frac{1}{2}u=\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\text{etc.}}{2\left(\frac{1}{2}+\frac{u^2}{2\cdot 4\cdot 6}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\text{etc.}\right)}$$

or

$$V-\frac{1}{2}u=\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^6}{2\cdot 4\cdots 12}+\frac{u^8}{2\cdot 4\cdots 16}+\text{etc.}}{1+\frac{u^2}{4\cdot 6}+\frac{u^4}{4\cdot 6\cdot 8\cdot 10}+\frac{u^6}{4\cdot 6\cdots 14}+\frac{u^8}{4\cdot 6\cdots 18}+\text{etc.}}$$

115. Since no odd powers occur in this fraction, likewise none can occur in its expansion; because $V-\frac{1}{2}u$ equals the series

$$1+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.},$$

the coefficients of the odd powers γ, ε, η, ι etc. all vanish. And so it is clear why the even-ordered terms after the second all equal zero in the series $1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+$ etc., for otherwise the law of continuity would be violated. Thus

$$V=1+\frac{1}{2}u+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\varkappa u^{10}+\text{etc.},$$

and if the letters β, δ, ζ, θ, $\varkappa$ have been determined by the development of the above fraction, one obtains the summative term Sz of the series, whose general term $=z$ corresponds to the index x, expressed as

$$Sz=\int zdx+\frac{1}{2}z+\frac{\beta dz}{dx}+\frac{\delta d^3z}{dx^3}+\frac{\zeta d^5z}{dx^5}+\frac{\theta d^7z}{dx^7}+\text{etc.}$$

116. Since the series $1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+$ etc. arises by developing the fraction

$$\frac{1+\frac{u^2}{2\cdot4}+\frac{u^4}{2\cdot4\cdot6\cdot8}+\frac{u^6}{2\cdot4\cdot6\cdot8\cdot10\cdot12}+\text{etc.}}{1+\frac{u^2}{4\cdot6}+\frac{u^4}{4\cdot6\cdot8\cdot10}+\frac{u^6}{4\cdot6\cdot8\cdot10\cdot12\cdot14}+\text{etc.}},$$

the letters β, δ, ζ, θ, x will follow according to the rule, as

$$\begin{aligned}
\beta &= \tfrac{1}{2\cdot4}-\tfrac{1}{4\cdot6}\\
\delta &= \tfrac{1}{2\cdot4\cdot6\cdot8}-\tfrac{\beta}{4\cdot6}-\tfrac{1}{4\cdot6\cdot8\cdot10}\\
\zeta &= \tfrac{1}{2\cdot4\cdot6\cdots12}-\tfrac{\delta}{4\cdot6}-\tfrac{\beta}{4\cdot6\cdot8\cdot10}-\tfrac{1}{4\cdot6\cdots14}\\
\theta &= \tfrac{1}{2\cdot4\cdot6\cdots16}-\tfrac{\zeta}{4\cdot6}-\tfrac{\delta}{4\cdot6\cdot8\cdot10}-\tfrac{\beta}{4\cdot6\cdots14}-\tfrac{1}{4\cdot6\cdots18}\\
&\text{etc.}
\end{aligned}$$

But these values are alternatingly positive and negative.

117. If the letters are alternatingly taken negatively, so that

$$Sz=\int zdx+\frac{1}{2}z-\frac{\beta dz}{dx}+\frac{\delta d^3z}{dx^3}-\frac{\zeta d^5z}{dx^5}+\frac{\theta d^7z}{dx^7}-\text{etc.},$$

then the letters β, δ, ζ, θ, etc. are determined by the fraction

$$\frac{1-\frac{u^2}{2\cdot4}+\frac{u^4}{2\cdot4\cdot6\cdot8}-\frac{u^6}{2\cdot4\cdots12}+\frac{u^8}{2\cdot4\cdots16}-\text{etc.}}{1-\frac{u^2}{4\cdot6}+\frac{u^4}{4\cdot6\cdot8\cdot10}-\frac{u^6}{4\cdot6\cdots14}+\frac{u^8}{4\cdot6\cdots18}-\text{etc.}},$$

when one develops it to the series

$$1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\text{etc.}$$

From this one has

$$\beta = \frac{1}{4\cdot 6} - \frac{1}{2\cdot 4}$$

$$\delta = \frac{\beta}{4\cdot 6} - \frac{1}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{2\cdot 4\cdot 6\cdot 8}$$

$$\zeta = \frac{\delta}{4\cdot 6} - \frac{\beta}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{4\cdot 6\cdots 14} - \frac{1}{2\cdot 4\cdots 12}$$

etc.;

only here all terms are negative.

118. Thus we consider $\beta = -A$; $\delta = -B$; $\zeta = -C$, etc.; consequently

$$Sz = \int z dx + \frac{1}{2}z + \frac{Adz}{dx} - \frac{Bd^3z}{dx^3} + \frac{Cd^5z}{dx^5} - \frac{Dd^7z}{dx^7} + \text{etc.},$$

and in order to determine the letters A, B, C, D etc., we consider the series

$$1 - Au^2 - Bu^4 - Cu^6 - Du^8 - Eu^{10} - \text{etc.},$$

which arises from the development of the fraction

$$\frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \frac{u^8}{2\cdot 4\cdots 16} - \text{etc.}}{1 - \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} - \frac{u^6}{4\cdot 6\cdots 14} + \frac{u^8}{4\cdot 6\cdots 18} - \text{etc.}},$$

or consider the series

$$\frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \text{etc.} \; = s,$$

which arises from the development of the fraction

$$s = \frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \text{etc.}}{u - \frac{u^3}{4\cdot 6} + \frac{u^5}{4\cdot 6\cdot 8\cdot 10} - \frac{u^7}{4\cdot 6\cdots 14} + \text{etc.}}.$$

But since

$$\cos\frac{1}{2}u = 1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \text{etc.},$$

$$\sin\frac{1}{2}u = \frac{u}{2} - \frac{u^3}{2\cdot 4\cdot 6} + \frac{u^5}{2\cdot 4\cdot 6\cdot 8\cdot 10} - \frac{u^7}{2\cdot 4\cdots 14} + \text{etc.},$$

we have

$$s = \frac{\cos\frac{1}{2}u}{2\sin\frac{1}{2}u} = \frac{1}{2}\cot\frac{1}{2}u.$$

Thus if one converts the cotangent of the arc $\frac{1}{2}u$ into a series, according to the powers of u, the values of the letters A, B, C, D, E, etc. are revealed.

119. Because $s = \frac{1}{2}\cot\frac{1}{2}u$, one has $\frac{1}{2}u = A.\cot 2s$, and if one differentiates, then $\frac{1}{2}du = \frac{-2ds}{1+4ss}$, or $4ds + du + 4ssdu = 0$, or

$$\frac{4ds}{du} + 1 + 4ss = 0.$$

But since

$$s = \frac{1}{u} - Au - Bu^3 - Cu^5 - \text{etc.},$$

one has

$$\begin{aligned}
\tfrac{4ds}{du} &= -\tfrac{4}{uu} - 4A - 3\cdot 4Bu^2 - 5\cdot 4Cu^4 - 7\cdot 4Du^6 - \text{etc.}\\
1 &= \quad\;\; 1\\
4ss &= \quad \tfrac{4}{uu} - 8A - \quad 8Bu^2 - \quad 8Cu^4 - \quad 8Du^6 - \text{etc.}\\
&\qquad\qquad\quad + \quad 4A^2u^2 + 8ABu^4 + 8ACu^6 + \text{etc.}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + 4BBu^6 + \text{etc.}
\end{aligned}$$

Setting the homogeneous terms to zero, one obtains

$$A = \frac{1}{12},\quad B = \frac{A^2}{5},\quad C = \frac{2AB}{7},\quad D = \frac{2AC+BB}{9},\quad E = \frac{2AD+2BC}{11},$$
$$F = \frac{2AE+2BD+CC}{13},\quad G = \frac{2AF+2BE+2CD}{15},$$
$$H = \frac{2AG+2BF+2CE+DD}{17},\quad \text{etc.}$$

From these formulas it is very clearly apparent that each of these values is positive .

120. But because the denominators of these fractions become very large, and substantially impede calculation, we want instead of the letters A, B, C, D, etc. to introduce new ones[1]:

$$A = \frac{\alpha}{1\cdot 2\cdot 3},\quad B = \frac{\beta}{1\cdot 2\cdot 3\cdot 4\cdot 5},\quad C = \frac{\gamma}{1\cdot 2\cdot 3\cdots 7},$$
$$D = \frac{\delta}{1\cdot 2\cdot 3\cdots 9},\quad E = \frac{\varepsilon}{1\cdot 2\cdot 3\cdots 11},\quad \text{etc.}$$

[1] Translator's note: **Caution!** These new symbols α, β, ... are **completely different** from the α, β, ... used earlier.

Then one finds

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2}{3}\alpha^2, \quad \gamma = 2\cdot\frac{3}{3}\alpha\beta, \quad \delta = 2\cdot\frac{4}{3}\alpha\gamma + \frac{8\cdot 7}{4\cdot 5}\beta^2,$$

$$\varepsilon = 2\cdot\frac{5}{3}\alpha\delta + 2\cdot\frac{10\cdot 9\cdot 8}{1\cdot 2\cdots 5}\beta\gamma,$$

$$\zeta = 2\cdot\frac{12}{1\cdot 2\cdot 3}\alpha\varepsilon + 2\cdot\frac{12\cdot 11\cdot 10}{1\cdot 2\cdots 5}\beta\delta + \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdots 7}\gamma\gamma,$$

$$\eta = 2\cdot\frac{14}{1\cdot 2\cdot 3}\alpha\zeta + 2\cdot\frac{14\cdot 13\cdot 12}{1\cdot 2\cdots 5}\beta\varepsilon + 2\cdot\frac{14\cdot 13\cdot 12\cdot 11\cdot 10}{1\cdot 2\cdots 7}\gamma\delta, \quad \text{etc.}$$

121. But it is more convenient to make use of the formulas

$$\alpha = \frac{1}{2}, \quad \beta = \frac{4}{3}\cdot\frac{\alpha\alpha}{2}, \quad \gamma = \frac{6}{3}\cdot\alpha\beta, \quad \delta = \frac{8}{3}\cdot\alpha\gamma + \frac{8\cdot 7\cdot 6}{3\cdot 4\cdot 5}\cdot\frac{\beta\beta}{2},$$

$$\varepsilon = \frac{10}{3}\cdot\alpha\delta + \frac{10\cdot 9\cdot 8}{3\cdot 4\cdot 5}\cdot\beta\gamma, \quad \zeta = \frac{12}{3}\cdot\alpha\varepsilon + \frac{12\cdot 11\cdot 10}{3\cdot 4\cdot 5}\cdot\beta\delta + \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\frac{\gamma\gamma}{2},$$

$$\eta = \frac{14}{3}\cdot\alpha\zeta + \frac{14\cdot 13\cdot 12}{3\cdot 4\cdot 5}\cdot\beta\varepsilon + \frac{14\cdot 13\cdot 12\cdot 11\cdot 10}{3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\gamma\delta,$$

$$\theta = \frac{16}{3}\cdot\alpha\eta + \frac{16\cdot 15\cdot 14}{3\cdot 4\cdot 5}\cdot\beta\zeta + \frac{16\cdot 15\cdots 12}{3\cdot 4\cdots 7}\cdot\gamma\varepsilon + \frac{16\cdot 15\cdots 10}{3\cdot 4\cdots 9}\cdot\frac{\delta\delta}{2}$$

etc.

If one finds the values of the letters α, β, γ, δ, etc. according to this rule, which entails little difficulty in calculation, then one can express the summative term of any series, whose general term $= z$ corresponding to the index x, in the following fashion:

$$Sz = \int z dx + \frac{1}{2}z + \frac{\alpha dz}{1\cdot 2\cdot 3\cdot dx} - \frac{\beta d^3 z}{1\cdot 2\cdot 3\cdot 4\cdot 5 dx^3} + \frac{\gamma d^5 z}{1\cdot 2\cdots 7 dx^5} - \frac{\delta d^7 z}{1\cdot 2\cdots 9 dx^7} + \frac{\varepsilon d^9 z}{1\cdot 2\cdots 11 dx^9} - \frac{\zeta d^{11} z}{1\cdot 2\cdots 13 dx^{11}} + \text{etc.}$$

As far as the letters α, β, γ, δ, etc. are concerned, one finds the following values:

$$
\begin{array}{lcrcl}
\alpha = \frac{1}{2} & \text{or} & 1\cdot 2\alpha &=& 1\\
\beta = \frac{1}{6} & & 1\cdot 2\cdot 3\beta &=& 1\\
\gamma = \frac{1}{6} & & 1\cdot 2\cdot 3\cdot 4\gamma &=& 4\\
\delta = \frac{3}{10} & & 1\cdot 2\cdot 3\cdots 5\delta &=& 36\\
\varepsilon = \frac{5}{6} & & 1\cdot 2\cdot 3\cdots 6\varepsilon &=& 600\\
\zeta = \frac{691}{210} & & 1\cdot 2\cdot 3\cdots 7\zeta &=& 24\cdot 691\\
\eta = \frac{35}{2} & & 1\cdot 2\cdot 3\cdots 8\eta &=& 20160\cdot 35\\
\theta = \frac{3617}{30} & & 1\cdot 2\cdot 3\cdots 9\theta &=& 12096\cdot 3617\\
\iota = \frac{43867}{42} & & 1\cdot 2\cdot 3\cdots 10\iota &=& 86400\cdot 43867\\
\varkappa = \frac{1222277}{110} & & 1\cdot 2\cdot 3\cdots 11\varkappa &=& 362880\cdot 1222277\\
\lambda = \frac{854513}{6} & & 1\cdot 2\cdot 3\cdots 12\lambda &=& 79833600\cdot 854513\\
\mu = \frac{1181820455}{546} & & 1\cdot 2\cdot 3\cdots 13\mu &=& 11404800\cdot 1181820455\\
\nu = \frac{76977927}{2} & & 1\cdot 2\cdot 3\cdots 14\nu &=& 43589145600\cdot 76977927\\
\xi = \frac{23749461029}{30} & & 1\cdot 2\cdot 3\cdots 15\xi &=& 43589145600\cdot 23749461029\\
\pi = \frac{8615841276005}{462} & & 1\cdot 2\cdot 3\cdots 16\pi &=& 45287424000\cdot 8615841276005
\end{array}
$$

etc.

122. These numbers have great use throughout the entire theory of series. First, one can obtain from them the final terms in the sums of even powers, for which we noted above (in §63 of part one) that one cannot obtain them, as one can the other terms, from the sums of earlier powers. For the even powers, the last terms of the sums are products of x and certain numbers, namely for the 2nd, 4th, 6th, 8th, etc., $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$ etc. with alternating signs. But these numbers arise from the values of the letters α, β, γ, δ, etc., which we found earlier, when one divides them by the odd numbers 3, 5, 7, 9, etc. These numbers are called the Bernoulli numbers after their discoverer Jakob Bernoulli,

and they are

$$\begin{array}{llll}
\frac{\alpha}{3}=\frac{1}{6}=\mathfrak{A} & \frac{\iota}{19}=\frac{43867}{798} & =\mathfrak{I} & \\
\frac{\beta}{5}=\frac{1}{30}=\mathfrak{B} & \frac{\varkappa}{21}=\frac{174611}{330} & =\mathfrak{K} & =\frac{283\cdot 617}{330} \\
\frac{\gamma}{7}=\frac{1}{42}=\mathfrak{C} & \frac{\lambda}{23}=\frac{854513}{138} & =\mathfrak{L} & =\frac{11\cdot 131\cdot 593}{2\cdot 3\cdot 23} \\
\frac{\delta}{9}=\frac{1}{30}=\mathfrak{D} & \frac{\mu}{25}=\frac{236364091}{2730} & =\mathfrak{M} & \\
\frac{\varepsilon}{11}=\frac{5}{66}=\mathfrak{E} & \frac{\nu}{27}=\frac{8553103}{6} & =\mathfrak{N} & =\frac{13\cdot 657931}{6} \\
\frac{\zeta}{13}=\frac{691}{2730}=\mathfrak{F} & \frac{\xi}{29}=\frac{23749461029}{870} & =\mathfrak{O} & \\
\frac{\eta}{15}=\frac{7}{6}=\mathfrak{G} & \frac{\pi}{31}=\frac{8615841276005}{14322} & =\mathfrak{P} & \\
\frac{\theta}{17}=\frac{3617}{510}=\mathfrak{H} & \text{etc.} & &
\end{array}$$

123. Thus one immediately obtains the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. from the following equations:

$$\begin{aligned}
\mathfrak{A} &= \frac{1}{6} \\
\mathfrak{B} &= \frac{4\cdot 3}{1\cdot 2}\cdot\frac{1}{5}\,\mathfrak{A}^2 \\
\mathfrak{C} &= \frac{6\cdot 5}{1\cdot 2}\cdot\frac{2}{7}\,\mathfrak{A}\mathfrak{B} \\
\mathfrak{D} &= \frac{8\cdot 7}{1\cdot 2}\cdot\frac{2}{9}\,\mathfrak{A}\mathfrak{C}+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{9}\mathfrak{B}^2 \\
\mathfrak{E} &= \frac{10\cdot 9}{1\cdot 2}\cdot\frac{2}{11}\,\mathfrak{A}\mathfrak{D}+\frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{2}{11}\mathfrak{B}\mathfrak{C} \\
\mathfrak{F} &= \frac{12\cdot 11}{1\cdot 2}\cdot\frac{2}{13}\,\mathfrak{A}\mathfrak{E}+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{2}{13}\mathfrak{B}\mathfrak{D}+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{1}{13}\,\mathfrak{C}^2 \\
\mathfrak{G} &= \frac{14\cdot 13}{1\cdot 2}\cdot\frac{2}{15}\,\mathfrak{A}\mathfrak{F}+\frac{14\cdot 13\cdot 12\cdot 11}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{2}{15}\mathfrak{B}\mathfrak{E}+\frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{2}{15}\,\mathfrak{C}\mathfrak{D} \\
&\text{etc.,}
\end{aligned}$$

and the law of these equations is clear if one notes that whenever the square of a letter appears, its coefficient is only half as large as it would appear according to the rule. But actually one should view the terms containing products of different letters as occurring twice. So for example

$$13\mathfrak{F}=\frac{12\cdot 11}{1\cdot 2}\mathfrak{A}\mathfrak{E}+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\mathfrak{B}\mathfrak{D}+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\mathfrak{C}\mathfrak{C}$$
$$+\frac{12\cdot 11\cdot 10\cdots 5}{1\cdot 2\cdot 3\cdots 8}\mathfrak{D}\mathfrak{B}+\frac{12\cdot 11\cdot 10\cdots 3}{1\cdot 2\cdot 3\cdots 10}\mathfrak{E}\mathfrak{A}.$$

124. Next, the numbers α, β, γ, δ etc. are also ingredients in the expressions for the sums of the series of fractions comprised by the general formula

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.},$$

when n is a positive even number. We expressed the sums of these series in the *Introductio*[2] via powers of the semiperiphery π of the circle of radius $=1$, and there one encounters the numbers α, β, γ, δ, etc. in the coefficients of

[2] Introductio, Book I, chapter 10.

these powers. But because these do not appear to occur by accident, rather their necessity is apparent, we wish to investigate these sums in a special way, by which the truth of the law of these sums will be clear. Because from above (§43) one has

$$\frac{\pi}{n}\cot\frac{m}{n}\pi = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.},$$

combining terms in pairs one has,

$$\frac{\pi}{n}\cot\frac{m}{n}\pi = \frac{1}{m} - \frac{2m}{nn-m^2} - \frac{2m}{4n^2-m^2} - \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2} - \text{etc.},$$

and from this

$$\frac{1}{n^2-m^2} + \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} + \frac{1}{16n^2-m^2} + \text{etc.} \;= \frac{1}{2mm} - \frac{\pi}{2mn}\cot\frac{m}{n}\pi.$$

Now we set $n = 1$ and replace m by u, yielding

$$\frac{1}{1-u^2} + \frac{1}{4-u^2} + \frac{1}{9-u^2} + \frac{1}{16-u^2} + \text{etc.} \;= \frac{1}{2uu} - \frac{\pi}{2u}\cot\pi u.$$

Resolving these fractions in series, one obtains

$$\begin{aligned}
\tfrac{1}{1-u^2} &= 1 + u^2 + u^4 + u^6 + u^8 + \text{etc.}\\
\tfrac{1}{4-u^2} &= \tfrac{1}{2^2} + \tfrac{u^2}{2^4} + \tfrac{u^4}{2^6} + \tfrac{u^6}{2^8} + \tfrac{u^8}{2^{10}} + \text{etc.}\\
\tfrac{1}{9-u^2} &= \tfrac{1}{3^2} + \tfrac{u^2}{3^4} + \tfrac{u^4}{3^6} + \tfrac{u^6}{3^8} + \tfrac{u^8}{3^{10}} + \text{etc.}\\
\tfrac{1}{16-u^2} &= \tfrac{1}{4^2} + \tfrac{u^2}{4^4} + \tfrac{u^4}{4^6} + \tfrac{u^6}{4^8} + \tfrac{u^8}{4^{10}} + \text{etc.}
\end{aligned}$$

125. If we thus set

$$\begin{array}{ll}
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.} = \mathfrak{a} & 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \text{etc.} = \mathfrak{d}\\
1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = \mathfrak{b} & 1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \text{etc.} = \mathfrak{e}\\
1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \text{etc.} = \mathfrak{c} & 1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \text{etc.} = \mathfrak{f}
\end{array}$$

etc.,

then the series above is transformed into

$$\mathfrak{a} + \mathfrak{b}u^2 + \mathfrak{c}u^4 + \mathfrak{d}u^6 + \mathfrak{e}u^8 + \mathfrak{f}u^{10} + \text{etc.} = \frac{1}{2uu} - \frac{\pi}{2u}\cot\pi u.$$

Now in §118 we found that for the letters A, B, C, D etc., when one sets

$$s = \frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \text{etc.}$$

one has $s = \frac{1}{2}\cot\frac{1}{2}u$, and thus, when one replaces $\frac{1}{2}u$ by πu, or u by $2\pi u$, one obtains

$$\frac{1}{2}\cot\pi u = \frac{1}{2\pi u} - 2A\pi u - 2^3 B\pi^3 u^3 - 2^5 C\pi^5 u^5 - 2^7 D\pi^7 u^7 - \text{etc.},$$

and multiplying by $\frac{\pi}{u}$ yields

$$\frac{\pi}{2u}\cot\pi u = \frac{1}{2uu} - 2A\pi^2 - 2^3 B\pi^4 u^2 - 2^5 C\pi^6 u^4 - 2^7 D\pi^8 u^6 - \text{etc.},$$

from which follows

$$\frac{1}{2uu} - \frac{\pi}{2u}\cot\pi u = 2A\pi^2 + 2^3 B\pi^4 u^2 + 2^5 C\pi^6 u^4 + 2^7 D\pi^8 u^6 + \text{etc.}$$

Since we already found that

$$\frac{1}{2uu} - \frac{\pi}{2u}\cot\pi u = \mathfrak{a} + \mathfrak{b}u^2 + \mathfrak{c}u^4 + \mathfrak{d}u^6 + \text{etc.},$$

it necessarily follows that

$$\begin{aligned}
\mathfrak{a} &= 2\ A\pi^2 = \tfrac{2\alpha}{1\cdot2\cdot3}\ \pi^2 = \tfrac{2\mathfrak{A}}{1\cdot2}\ \pi^2 \\
\mathfrak{b} &= 2^3\ B\pi^4 = \tfrac{2^3\beta}{1\cdot2\cdot3\cdot4\cdot5}\ \pi^4 = \tfrac{2^3\mathfrak{B}}{1\cdot2\cdot3\cdot4}\ \pi^4 \\
\mathfrak{c} &= 2^5\ C\pi^6 = \tfrac{2^5\gamma}{1\cdot2\cdot3\cdots7}\ \pi^6 = \tfrac{2^5\mathfrak{C}}{1\cdot2\cdots6}\ \pi^6 \\
\mathfrak{d} &= 2^7\ D\pi^8 = \tfrac{2^7\delta}{1\cdot2\cdot3\cdots9}\ \pi^8 = \tfrac{2^7\mathfrak{D}}{1\cdot2\cdots8}\ \pi^8 \\
\mathfrak{e} &= 2^9\ E\pi^{10} = \tfrac{2^9\varepsilon}{1\cdot2\cdot3\cdots11}\ \pi^{10} = \tfrac{2^9\mathfrak{E}}{1\cdot2\cdots10}\ \pi^{10} \\
\mathfrak{f} &= 2^{11}\ F\pi^{12} = \tfrac{2^{11}\zeta}{1\cdot2\cdot3\cdots13}\ \pi^{12} = \tfrac{2^{11}\mathfrak{F}}{1\cdot2\cdots12}\ \pi^{12} \\
&\text{etc.}
\end{aligned}$$

[...]

129. From the table of values of the numbers α, β, γ, δ etc. that we communicated above in §121, it is apparent that they at first decrease, but then grow without end. Thus it is worth the effort to investigate in what ratio these numbers continue to grow, after they reach considerable size. So let φ be a number far from the beginning in the sequence α, β, γ, δ, etc., and ψ the one immediately following. Since the sums of the reciprocal powers are determined by these numbers, we let $2n$ be the exponent of the power, in whose sum φ occurs; $2n+2$ will be the exponent of the power corresponding to ψ, and n a very large number. Then from §125 one has

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \text{etc.} = \frac{2^{2n-1}\varphi}{1\cdot2\cdot3\cdots(2n+1)}\pi^{2n},$$

$$1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \frac{1}{4^{2n+2}} + \text{etc.} = \frac{2^{2n+1}\psi}{1\cdot2\cdot3\cdots(2n+3)}\pi^{2n+2}.$$

Dividing this series by the former, one finds

$$\frac{1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \text{etc.}}{1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \text{etc.}} = \frac{4\psi\pi^2}{(2n+2)\,(2n+3)\,\varphi}.$$

But because n is a very large number and both series are very closely $= 1$,

$$\frac{\psi}{\varphi} = \frac{(2n+2)\,(2n+3)}{4\pi^2} = \frac{nn}{\pi\pi}.$$

Now n indicates which term the number φ is beyond the first number α, and from this the number φ is to the following ψ as π^2 is to n^2, and this ratio would, if n were an infinitely large number, be in complete accordance with the truth. Because $\pi\pi$ nearly $= 10$, when one lets $n = 100$, the hundredth term is approximately 1000 times smaller than the subsequent one. Thus the numbers α, β, γ, δ etc., and also the Bernoullian $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., form a highly diverging sequence, which grows more strongly than any geometric sequence of growing terms.

130. Thus if one has found the numbers α, β, γ, δ etc., or $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., then given a series, whose general term z is a function of its index x, the summative term Sz can be expressed as follows:

$$\begin{aligned} Sz = \int z dx + \frac{1}{2} z + \frac{1}{6} \cdot \frac{dz}{1 \cdot 2 dx} - \frac{1}{30} \cdot \frac{d^3 z}{1 \cdot 2 \cdot 3 \cdot 4 dx^3} \\ + \frac{1}{42} \cdot \frac{d^5 z}{1 \cdot 2 \cdot 3 \cdots 6 dx^5} - \frac{1}{30} \cdot \frac{d^7 z}{1 \cdot 2 \cdot 3 \cdots 8 dx^7} \\ + \frac{5}{66} \cdot \frac{d^9 z}{1 \cdot 2 \cdot 3 \cdots 10 dx^9} - \frac{691}{2730} \cdot \frac{d^{11} z}{1 \cdot 2 \cdot 3 \cdots 12 dx^{11}} \\ + \frac{7}{6} \cdot \frac{d^{13} z}{1 \cdot 2 \cdot 3 \cdots 14 dx^{13}} - \frac{3617}{510} \cdot \frac{d^{15} z}{1 \cdot 2 \cdot 3 \cdots 16 dx^{15}} \\ + \frac{43867}{798} \cdot \frac{d^{17} z}{1 \cdot 2 \cdot 3 \cdots 18 dx^{17}} - \frac{174611}{330} \cdot \frac{d^{19} z}{1 \cdot 2 \cdot 3 \cdots 20 dx^{19}} \\ + \frac{854513}{138} \cdot \frac{d^{21} z}{1 \cdot 2 \cdot 3 \cdots 22 dx^{21}} - \frac{236364091}{2730} \cdot \frac{d^{23} z}{1 \cdot 2 \cdot 3 \cdots 24 dx^{23}} \\ + \frac{8553103}{6} \cdot \frac{d^{25} z}{1 \cdot 2 \cdot 3 \cdots 26 dx^{25}} - \frac{23749461029}{870} \cdot \frac{d^{27} z}{1 \cdot 2 \cdot 3 \cdots 28 dx^{27}} \\ + \frac{8615841276005}{14322} \cdot \frac{d^{29} z}{1 \cdot 2 \cdot 3 \cdots 30 dx^{29}} - \text{etc.} \end{aligned}$$

Thus if one knows the integral $\int z dx$, or the quantity, whose differential is $= z dx$, one finds the summative term by means of continuing differentiation. One must not neglect that a constant value must always be added to this expression, of a nature that the sum will $= 0$, when x becomes 0.

131. If now z is an integral rational function of x, so that the derivatives eventually vanish, then the summative term is represented by a finite expression. We illustrate this by some examples.

First example.

Find the summative term of the following series.

$$\begin{array}{l} 1 \quad 2 \quad\; 3 \quad\;\; 4 \quad\;\; 5 \qquad\qquad\qquad\quad x \\ 1+9+25+49+81+\cdots+(2x-1)^2 \end{array}$$

Since here $z=(2x-1)^2=4xx-4x+1$, one has

$$\int zdx=\frac{4}{3}x^3-2x^2+x;$$

because from this, differentiation produces $4xxdx-4xdx+dx=zdx$. Further differentiation yields

$$\frac{dz}{dx}=8x-4, \quad \frac{ddz}{dx^2}=8, \quad \frac{d^3z}{dx^3}=0 \quad \text{etc.}$$

So the summative term sought equals

$$\frac{4}{3}x^3-2x^2+x+2xx-2x+\frac{1}{2}+\frac{2}{3}x-\frac{1}{3}\pm\text{Const.},$$

in which the constant must remove the terms $\frac{1}{2}-\frac{1}{3}$, so

$$S\,(2x-1)^2=\frac{4}{3}x^3-\frac{1}{3}x=\frac{x}{3}\,(2x-1)\,(2x+1)\,.$$

So if one sets $x=4$, the sum of the first four terms

$$1+9+25+49=\frac{4}{3}\cdot 7\cdot 9=84.$$

[...]

132. From this general expression for the summative term, the sum for powers of natural numbers, that we communicated in the first part (§29 and 61), but which we could not prove at that time, follows very easily. Let us set $z=x^n$, so that $\int zdx=\frac{1}{n+1}x^{n+1}$, and differentiating,

$$\frac{dz}{dx}=nx^{n-1},\ \frac{ddz}{dx^2}=n\,(n-1)\,x^{n-2},\ \frac{d^3z}{dx^3}=n\,(n-1)\,(n-2)\,x^{n-3},$$
$$\frac{d^5z}{dx^5}=n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,x^{n-5},\ \frac{d^7z}{dx^7}=n\,(n-1)\cdots(n-6)\,x^{n-7},\text{etc.}$$

From this we deduce the following summative term corresponding to the general term x^n:

$$
\begin{aligned}
Sx^n = \frac{1}{n+1}x^{n+1} + \frac{1}{2}x^n + \frac{1}{6}\cdot\frac{n}{2}x^{n-1} - \frac{1}{30}\cdot\frac{n\,(n-1)\,(n-2)}{2\cdot 3\cdot 4}x^{n-3} \\
+\frac{1}{42}\cdot\frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^{n-5} \\
-\frac{1}{30}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-6)}{2\cdot 3\cdots 8}x^{n-7} \\
+\frac{5}{66}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-8)}{2\cdot 3\cdots 10}x^{n-9} \\
-\frac{691}{2730}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-10)}{2\cdot 3\cdots 12}x^{n-11} \\
+\frac{7}{6}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-12)}{2\cdot 3\cdots 14}x^{n-13} \\
-\frac{3617}{510}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-14)}{2\cdot 3\cdots 16}x^{n-15} \\
+\frac{43867}{798}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-16)}{2\cdot 3\cdots 18}x^{n-17} \\
-\frac{174611}{330}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-18)}{2\cdot 3\cdots 20}x^{n-19} \\
+\frac{854513}{138}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-20)}{2\cdot 3\cdots 22}x^{n-21} \\
-\frac{236364091}{2730}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-22)}{2\cdot 3\cdots 24}x^{n-23} \\
+\frac{8553103}{6}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-24)}{2\cdot 3\ldots 26}x^{n-25} \\
-\frac{23749461029}{870}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-26)}{2\cdot 3\cdots 28}x^{n-27} \\
+\frac{8615841276005}{14322}\cdot\frac{n\,(n-1)\cdots\cdots\cdots(n-28)}{2\cdot 3\cdots 30}x^{n-29} \\
\text{etc.}
\end{aligned}
$$

This expression differs from the former only in that here we have introduced the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., whereas above we used the numbers α, β, γ, δ etc.; the agreement is clear. Thus here we have been able to give the summative terms for all powers up to the thirtieth, inclusive; if we wanted to perform this investigation via other means, lengthy and tedious calculations would be necessary.

[...]

Part Two, Chapter 6
On the summing of progressions via infinite series

140. The general expression, that we found in the previous chapter for the summative term of a series, whose general term corresponding to the index x

is z, namely

$$Sz = \int z dx + \frac{1}{2} z + \frac{\mathfrak{A} dz}{1 \cdot 2 dx} - \frac{\mathfrak{B} d^3 z}{1 \cdot 2 \cdot 3 \cdot 4 dx^3} + \frac{\mathfrak{C} d^5 z}{1 \cdot 2 \cdots 6 dx^5} - \text{etc.},$$

actually serves to determine the sums of series, whose general terms are integral rational functions of the index x, because in these cases one eventually arrives at vanishing differentials. On the other hand, if z is not such a function of x, then the differentials continue without end, and there results an infinite series that expresses the sum of the given series up to and including the term whose index $= x$. The sum of the series, continuing without end, is thus given by taking $x = \infty$, and one finds in this way another infinite series equal to the original.

141. If one sets $x = 0$, the expression represented by the series must vanish, as we already noted; and if this does not occur, one must add to or take away from the sum a constant amount, so that this requirement is satisfied. If this is the case, then when $x = 1$ one obtains the first term of the series, when $x = 2$ the sum of the first and second, when $x = 3$ the sum of the first three terms of the series, etc. Because in these cases the sum of the first, first two, first three, etc. terms is known, this is also the value of the infinite series expressing the sum; and thus one is placed in a position to sum countlessly many series.

142. Since when a constant value is added to the sum, so that it vanishes when $x = 0$, the true sum is then found when x is any other number, then it is clear that the true sum must likewise be given, whenever a constant value is added that produces the true sum in any particular case. Thus suppose it is not obvious, when one sets $x = 0$, what value the sum assumes and thus what constant must be used; one can substitute other values for x, and through addition of a constant value obtain a complete expression for the sum. Much will become clear from the following.

142a. Consider first the harmonic progression

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x} = s.$$

Since the general term $= \frac{1}{x}$, we have $z = \frac{1}{x}$, and the summative term s will be found as follows. First one has $\int z dx = \int \frac{dx}{x} = lx$; from differentiation one has

$$\frac{dz}{dx} = -\frac{1}{x^2},\ \frac{ddz}{2dx^2} = \frac{1}{x^3},\ \frac{d^3 z}{6dx^3} = -\frac{1}{x^4},\ \frac{d^4 z}{24dx^4} = \frac{1}{x^5},\ \frac{d^5 z}{120dx^5} = -\frac{1}{x^6},\ \text{etc.}$$

From this

$$s = lx + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{ etc.} + \text{Constant}.$$

However, the added constant value cannot be determined from the case when $x = 0$. So we set $x = 1$. Since then $s = 1$, one has

$$1 = \frac{1}{2} - \frac{\mathfrak{A}}{2} + \frac{\mathfrak{B}}{4} - \frac{\mathfrak{C}}{6} + \frac{\mathfrak{D}}{8} - \text{etc.} + \text{Constant},$$

and thus the constant is

$$= \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \text{etc.}$$

Consequently the summative term sought is

$$\begin{aligned} s = lx &+ \tfrac{1}{2x} - \tfrac{\mathfrak{A}}{2x^2} + \tfrac{\mathfrak{B}}{4x^4} - \tfrac{\mathfrak{C}}{6x^6} + \tfrac{\mathfrak{D}}{8x^8} - \text{etc.} \\ &+ \tfrac{1}{2} + \tfrac{\mathfrak{A}}{2} - \tfrac{\mathfrak{B}}{4} + \tfrac{\mathfrak{C}}{6} - \tfrac{\mathfrak{D}}{8} + \text{etc.} \end{aligned}$$

143. Since the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. form a diverging series, it is not possible to really know the value of the constant here. But if we substitute a larger number for x, and really find the sum of that many terms, the value of the constant can be found easily. Let us set as the end $x = 10$; the sum of the first ten terms

$$= 2,928968253968253968,$$

which must equal the expression for the sum when one sets $x = 10$ in it, yielding

$$l10 + \frac{1}{20} - \frac{\mathfrak{A}}{200} + \frac{\mathfrak{B}}{40000} - \frac{\mathfrak{C}}{6000000} + \frac{\mathfrak{D}}{800000000} - \text{etc.} + C.$$

Thus if one substitutes for $l10$ the hyperbolic logarithm of 10, and in place of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. substitutes the values found above [§122], one obtains for the constant

$$C = 0,5772156649015325,$$

and this number therefore expresses the sum of the series

$$\frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \frac{\mathfrak{E}}{10} - \text{etc.}$$

144. If one substitutes for x a not very large number, then the sum of the [original] series is easy to find, and one obtains the sum

$$\frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{etc.} = s - lx - C.$$

But if x is a very large number, then the sum of this infinite expression can be found in decimal fractions. Now it is clear to begin with that if the [original] series continues infinitely, the sum will have infinite magnitude, because as $x = \infty$, also lx grows to infinity. But in order nonetheless to be able to give the sum of any number of terms more easily, we express the values of the letters $\mathfrak{A}$,

$\mathfrak{B}$, $\mathfrak{C}$ etc. in decimal fractions.

$$\begin{aligned}
\mathfrak{A} &= 0,1666666666666\\
\mathfrak{B} &= 0,0333333333333\\
\mathfrak{C} &= 0,0238095238095\\
\mathfrak{D} &= 0,0333333333333\\
\mathfrak{E} &= 0,0757575757575\\
\mathfrak{F} &= 0,2531135531135\\
\mathfrak{G} &= 1,1666666666666\\
\mathfrak{H} &= 7,0921568627451 \text{ etc.,}
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\mathfrak{A}}{2} &= 0,0833333333333\\
\frac{\mathfrak{B}}{4} &= 0,0083333333333\\
\frac{\mathfrak{C}}{6} &= 0,0039682539682\\
\frac{\mathfrak{D}}{8} &= 0,0041666666666\\
\frac{\mathfrak{E}}{10} &= 0,0075757575757\\
\frac{\mathfrak{F}}{12} &= 0,0210927960928\\
\frac{\mathfrak{G}}{14} &= 0,0833333333333\\
\frac{\mathfrak{H}}{16} &= 0,4432598039216 \text{ etc.}
\end{aligned}$$

First example.

Find the sum of one thousand terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ etc.

Set $x = 1000$, and because

$$\begin{aligned}
l10 &= 2,3025850929940456840, \\
\text{one has } lx &= 6,9077552789821 \\
\text{Const.} &= 0,5772156649015 \\
\frac{1}{2x} &= \underline{0,0005000000000} \\
& \quad\ 7,4854709438836 \\
\text{subtr. } \frac{\mathfrak{A}}{2xx} &= \underline{0,0000000833333} \\
& \quad\ 7,4854708605503 \\
\text{add } \frac{\mathfrak{B}}{4x^4} &= \underline{0,0000000000000,} \\
\text{thus} \qquad & \quad\ 7,4854708605503
\end{aligned}$$

is the desired sum of a thousand terms, which is still not seven and a half.

[...]

148. After considering the harmonic series we wish to turn to examining the series of reciprocals of the squares, letting

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{xx}.$$

Since the general term of this series is $z = \frac{1}{xx}$, then $\int z dx = \frac{1}{x}$, the differentials of z are

$$\frac{dz}{2dx} = -\frac{1}{x^3}, \quad \frac{ddz}{2 \cdot 3dx^2} = \frac{1}{x^4}, \quad \frac{d^3z}{2 \cdot 3 \cdot 4dx^3} = -\frac{1}{x^5} \quad \text{etc.},$$

and the sum is

$$s = C - \frac{1}{x} + \frac{1}{2xx} - \frac{\mathfrak{A}}{x^3} + \frac{\mathfrak{B}}{x^5} - \frac{\mathfrak{C}}{x^7} + \frac{\mathfrak{D}}{x^9} - \frac{\mathfrak{E}}{x^{11}} + \text{etc.},$$

where the added constant C is determined from one case in which the sum is known. We therefore wish to set $x = 1$. Since then $s = 1$, one has

$$C = 1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{ etc.},$$

but this series alone does not give the value of C, since it diverges strongly. Above [§125] we demonstrated that the sum of the series to infinity is $= \frac{\pi\pi}{6}$, and therefore setting $x = \infty$, and $s = \frac{\pi\pi}{6}$, we have $C = \frac{\pi\pi}{6}$, because then all other terms vanish. Thus it follows that

$$1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{etc.} \ = \frac{\pi\pi}{6}.$$

149. If the sum of this series were not known, then one would need to determine the value of the constant C from another case, in which the sum

were actually found. To this aim we set $x = 10$ and actually add up ten terms, obtaining

$$
\begin{array}{llcl}
 & s & = & 1,549767731166540690. \\
\text{Further, add} & \frac{1}{x} & = & 0,1 \\
\text{subtr.} & \frac{1}{2xx} & = & \underline{0,005} \\
 & & & 1,644767731166540690 \\
\text{add} & \frac{\mathfrak{A}}{x^3} & = & \underline{0,000166666666666666} \\
 & & & 1,644934397833207356 \\
\text{subtr.} & \frac{\mathfrak{B}}{x^5} & = & \underline{0,000000333333333333} \\
 & & & 1,644934064499874023 \\
\text{add} & \frac{\mathfrak{C}}{x^7} & = & \underline{0,000000002380952381} \\
 & & & 1,644934066880826404 \\
\text{subtr.} & \frac{\mathfrak{D}}{x^9} & = & \underline{0,000000000033333333} \\
 & & & 1,644934066847493071 \\
\text{add} & \frac{\mathfrak{E}}{x^{11}} & = & \underline{0,000000000000757575} \\
 & & & 1,644934066848250646 \\
\text{subtr.} & \frac{\mathfrak{F}}{x^{13}} & = & \underline{0,000000000000025311} \\
 & & & 1,644934066848225335 \\
\text{add} & \frac{\mathfrak{G}}{x^{15}} & = & 0,000000000000001166 \\
\text{subtr.} & \frac{\mathfrak{H}}{x^{17}} & = & \underline{\hfill 71} \\
 & & & 1,644934066848226430 \quad = C.
\end{array}
$$

This number is likewise the value of the expression $\frac{\pi\pi}{6}$, as one can find by calculation from the known value of π. From this it is clear that, although the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. diverges, it nevertheless produces a true sum. [...]

In §150–153 Euler explores possible formulas for exact sums of the infinite series of reciprocal odd powers of the natural numbers, similar to those he has already found for the sums of reciprocal even powers in terms of the Bernoulli numbers and π. Using his summation formula he produces highly accurate decimal approximations for the sums of reciprocal odd powers all the way through the fifteenth power, hoping to see a pattern analogous to the even powers, namely simple fractions times the relevant power of π. He is disappointed, however, not to find that they behave similarly to the even powers.

Then in §154–156 Euler uses a sum and the inverse tangent and cotangent functions to approximate π to seventeen decimal places with his summation formula, and remarks that it is amazing that one can approximate π so accurately with such an easy calculation.

157. Now we want to use for z transcendental functions of x, and take $z = lx$ for summing hyperbolic logarithms, from which the ordinary can easily be recovered, so that

$$s = l1 + l2 + l3 + l4 + \cdots + lx.$$

Because $z = lx$,

$$\int z dx = xlx - x,$$

since its differential is $dxlx$. Then

$$\frac{dz}{dx} = \frac{1}{x},\ \frac{ddz}{dx^2} = -\frac{1}{x^2},\ \frac{d^3z}{1\cdot 2dx^3} = \frac{1}{x^3},\ \frac{d^4z}{1\cdot 2\cdot 3dx^4} = -\frac{1}{x^4},\ \frac{d^5z}{1\cdot 2\cdot 3\cdot 4dx^5} = \frac{1}{x^5},\ \text{etc.}$$

One concludes that

$$s = xlx - x + \frac{1}{2}lx + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.} + \text{Const.}$$

But for this constant one finds, when one sets $x = 1$, because then $s = l1 = 0$,

$$C = 1 - \frac{\mathfrak{A}}{1\cdot 2} + \frac{\mathfrak{B}}{3\cdot 4} - \frac{\mathfrak{C}}{5\cdot 6} + \frac{\mathfrak{D}}{7\cdot 8} - \text{etc.},$$

a series that, due to its great divergence, is quite unsuitable even for determining the approximate value of C.

158. Nevertheless we can not only approximate the correct value of C, but can obtain it exactly, by considering Wallis's expression for π provided in the *Introductio* [1, vol. 1, chap. 11]. This expression is

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot \text{etc.}}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot\ \text{etc.}}$$

Taking logarithms, one obtains from this

$$\begin{aligned} l\pi - l2 &= 2l2 + 2l4 + 2l6 + 2l8 + 2l10 + l12 + \text{etc.} \\ &\quad -l1 - 2l3 - 2l5 - 2l7 - 2l9 - 2l11 - \text{etc.} \end{aligned}$$

Setting $x = \infty$ in the assumed series, we have

$$\begin{aligned} & l1 + l2 + l3 + l4 + \cdots + lx = C + \left(x + \tfrac{1}{2}\right) lx - x, \\ \text{thus}\quad & l1 + l2 + l3 + l4 + \cdots + l2x = C + \left(2x + \tfrac{1}{2}\right) l2x - 2x \\ \text{and}\quad & l2 + l4 + l6 + l8 + \cdots + l2x = C + \left(x + \tfrac{1}{2}\right) lx + xl2 - x, \\ \text{and therefore}\quad & l1 + l3 + l5 + l7 + \cdots + l\left(2x - 1\right) = xlx + \left(x + \tfrac{1}{2}\right) l2 - x. \end{aligned}$$

Thus because

$$\begin{aligned} l\tfrac{\pi}{2} &= 2l2 + 2l4 + 2l6 + \cdots + 2l2x - l2x \\ &\quad - 2l1 - 2l3 - 2l5 - \cdots - 2l\left(2x - 1\right), \end{aligned}$$

letting $x = \infty$ yields

$$l\frac{\pi}{2} = 2C + (2x+1)\,lx + 2xl2 - 2x - l2 - lx - 2xlx - (2x+1)\,l2 + 2x,$$

and therefore

$$l\frac{\pi}{2} = 2C - 2l2, \text{thus } 2C = l2\pi \text{and } C = \frac{1}{2}l2\pi,$$

yielding the decimal fraction representation

$$C = 0,9189385332046727417803297,$$

thus simultaneously the sum of the series

$$1 - \frac{\mathfrak{A}}{1\cdot 2} + \frac{\mathfrak{B}}{3\cdot 4} - \frac{\mathfrak{C}}{5\cdot 6} + \frac{\mathfrak{D}}{7\cdot 8} - \frac{\mathfrak{E}}{9\cdot 10} + \text{etc.} = \frac{1}{2}l2\pi.$$

159. Since we now know the constant $C = \frac{1}{2}l2\pi$, one can exhibit the sum of any number of logarithms from the series $l1 + l2 + l3 +$ etc. If one sets

$$s = l1 + l2 + l3 + l4 + \cdots + lx,$$

then

$$s = \frac{1}{2}l2\pi + \left(x + \frac{1}{2}\right)lx - x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.}$$

if the proposed logarithms are hyperbolic; if however the proposed logarithms are common, then one must take common logarithms also in the terms $\frac{1}{2}l2\pi + (x+\frac{1}{2})lx$ for $l2\pi$ and lx, and multiply the remaining terms

$$-x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \text{etc.}$$

of the series by $0,434294481903251827 = n$. In this case the common logarithms are

$$\begin{aligned} l\pi &= 0,497149872694133854351268 \\ l2 &= \underline{0,301029995663981195213738} \\ l2\pi &= 0,798179868358115049565006 \\ \frac{1}{2}l2\pi &= 0,399089934179057524782503. \end{aligned}$$

Example.

Find the sum of the first thousand common logarithms

$$s = l1 + l2 + l3 + \cdots + l1000.$$

So $x = 1000$, and

$$\begin{array}{rr}
lx = & 3,0000000000000, \\
\text{and thus } xlx = & 3000,0000000000000 \\
\frac{1}{2}lx = & 1,5000000000000 \\
\frac{1}{2}l2\pi = & \underline{0,3990899341790} \\
& 3001,8990899341790 \\
\text{subtr. } nx = & \underline{434,2944819032518} \\
& 2567,6046080309272.
\end{array}$$

Then

$$\begin{array}{rr}
\frac{n\mathfrak{A}}{1\cdot 2x} = & 0,0000361912068 \\
\text{subtr. } \frac{n\mathfrak{B}}{3\cdot 4x^3} = & \underline{0,0000000000012} \\
& 0,0000361912056 \\
\text{add} & \underline{2567,6046080309272} \\
\text{the sum sought } s = & 2567,6046442221328.
\end{array}$$

Now because s is the logarithm of a product of numbers

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots 1000,$$

it is clear that this product, if one actually multiplies it out, consists of 2568 figures, beginning with the figures 4023872, with 2561 subsequent figures.

160. By means of this summation of logarithms, one can approximate the product of any number of factors, that progress in the order of the natural numbers. This can be especially helpful for the problem of finding the middle or largest coefficient of any power in the binomial $(a+b)^m$, where one notes that, when m is an odd number, one always has two equal middle coefficients, which taken together produce the middle coefficient of the next even power. Thus since the largest coefficient of any even power is twice as large as the middle coefficient of the immediately preceding odd power, it suffices to determine the middle largest coefficient of an even power. Thus we have $m = 2n$ with middle coefficient expressed as

$$\frac{2n\,(2n-1)\,(2n-2)\,(2n-3)\cdots(n+1)}{1\cdot 2\cdot 3\cdot 4\cdots n}.$$

Setting this $= u$, one has

$$u = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots 2n}{(1\cdot 2\cdot 3\cdot 4\cdots n)^2},$$

and taking logarithms

$$\begin{gathered}
lu = l1 + l2 + l3 + l4 + l5 + \cdots\ l2n \\
-2l1 - 2l2 - 2l3 - 2l4 - 2l5 - \cdots - 2ln.
\end{gathered}$$

161. The sum of hyperbolic logarithms is

$$l1+l2+l3+l4+\cdots+l2n=\frac{1}{2}l2\pi+\left(2n+\frac{1}{2}\right)ln+\left(2n+\frac{1}{2}\right)l2-2n \\ +\frac{\mathfrak{A}}{1\cdot 2\cdot 2n}-\frac{\mathfrak{B}}{3\cdot 4\cdot 2^3n^3}+\frac{\mathfrak{C}}{5\cdot 6\cdot 2^5n^5}-\text{etc.}$$

and

$$2l1+2l2+2l3+2l4+\cdots+2ln \\ =l2\pi+(2n+1)\,ln-2n+\frac{2\mathfrak{A}}{1\cdot 2n}-\frac{2\mathfrak{B}}{3\cdot 4n^3}+\frac{2\mathfrak{C}}{5\cdot 6n^5}-\text{etc.}$$

Subtracting this expression from the former yields

$$lu=-\frac{1}{2}l\pi-\frac{1}{2}ln+2nl2+\frac{\mathfrak{A}}{1\cdot 2\cdot 2n}-\frac{\mathfrak{B}}{3\cdot 4\cdot 2^3n^3}+\frac{\mathfrak{C}}{5\cdot 6\cdot 2^5n^5}-\text{etc.} \\ -\frac{2\mathfrak{A}}{1\cdot 2n}+\frac{2\mathfrak{B}}{3\cdot 4n^3}-\frac{2\mathfrak{C}}{5\cdot 6n^5}+\text{etc.},$$

and collecting terms in pairs

$$lu=l\frac{2^{2n}}{\sqrt{n\pi}}-\frac{3\mathfrak{A}}{1\cdot 2\cdot 2n}+\frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3n^3}-\frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5n^5}+\frac{255\mathfrak{D}}{7\cdot 8\cdot 2^7n^7}-\text{etc.}$$

One has

$$\frac{3\mathfrak{A}}{1\cdot 2\cdot 2^2n^2}-\frac{15\mathfrak{B}}{3\cdot 4\cdot 2^4n^4}+\frac{63\mathfrak{C}}{5\cdot 6\cdot 2^6n^6}-\frac{255\mathfrak{D}}{7\cdot 8\cdot 2^8n^8}+\text{etc.} \\ =l\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\frac{D}{2^8n^8}+\text{etc.}\right),$$

so that

$$lu=l\frac{2^{2n}}{\sqrt{n\pi}}-2nl\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\text{etc.}\right)$$

and thus

$$u=\frac{2^{2n}}{\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\text{ etc.}\right)^{2n}\sqrt{n\pi}}.$$

Setting $2n = m$,

$$\begin{aligned}
&l(1 + \tfrac{A}{2^2n^2} + \tfrac{B}{2^4n^4} + \tfrac{C}{2^6n^6} + \tfrac{D}{2^8n^8} + \text{etc.}) \\
&= \tfrac{A}{m^2} + \tfrac{B}{m^4} + \tfrac{C}{m^6} + \tfrac{D}{m^8} + \tfrac{E}{m^{10}} + \text{etc.} \\
&\quad - \tfrac{A^2}{2m^4} - \tfrac{AB}{m^6} - \tfrac{AC}{m^8} - \tfrac{AD}{m^{10}} - \text{etc.} \\
&\quad - \tfrac{BB}{2m^8} - \tfrac{BC}{m^{10}} - \text{etc.} \\
&\quad + \tfrac{A^3}{3m^6} + \tfrac{A^2B}{m^8} + \tfrac{A^2C}{m^{10}} + \text{etc.} \\
&\quad + \tfrac{AB^2}{m^{10}} + \text{etc.} \\
&\quad - \tfrac{A^4}{4m^8} - \tfrac{A^3B}{m^{10}} - \text{etc.} \\
&\quad + \tfrac{A^5}{5m^{10}} + \text{etc.};
\end{aligned}$$

and because this expression must equal

$$\frac{3\mathfrak{A}}{1\cdot 2m^2} - \frac{15\mathfrak{B}}{3\cdot 4m^4} + \frac{63\mathfrak{C}}{5\cdot 6m^6} - \frac{255\mathfrak{D}}{7\cdot 8m^8} + \text{etc.},$$

one has

$$\begin{aligned}
A &= \frac{3\mathfrak{A}}{1\cdot 2} \\
B &= \frac{A^2}{2} - \frac{15\mathfrak{B}}{3\cdot 4} \\
C &= AB - \frac{1}{3}A^3 + \frac{63\mathfrak{C}}{5\cdot 6} \\
D &= AC + \frac{1}{2}B^2 - A^2B + \frac{1}{4}A^4 - \frac{255\mathfrak{D}}{7\cdot 8} \\
E &= AD + BC - A^2C - AB^2 + A^3B - \frac{1}{5}A^5 + \frac{1023\mathfrak{E}}{9\cdot 10} \\
&\qquad \text{etc.}
\end{aligned}$$

162. Now since $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$, $\mathfrak{D} = \frac{1}{30}$, $\mathfrak{E} = \frac{5}{66}$, one has

$$A = \frac{1}{4}, \quad B = -\frac{1}{96}, \quad C = \frac{27}{640}, \quad D = -\frac{90031}{2^{11}\cdot 3^2\cdot 5\cdot 7} \quad \text{etc.}$$

Consequently

$$u = \frac{2^{2n}}{\left(1 + \frac{1}{2^4n^2} - \frac{1}{2^9\cdot 3n^4} + \frac{27}{2^{13}\cdot 5n^6} - \frac{90031}{2^{19}\cdot 3^2\cdot 5\cdot 7n^8} + \text{etc.}\right)^{2n}\sqrt{n\pi}}$$

or

$$u = \frac{2^{2n}\left(1 - \frac{1}{2^4n^2} + \frac{7}{2^9\cdot 3n^4} - \frac{121}{2^{13}\cdot 3\cdot 5n^6} + \frac{107489}{2^{19}\cdot 3^2\cdot 5\cdot 7n^8} - \text{etc.}\right)^{2n}}{\sqrt{n\pi}},$$

or, if one actually takes the power of the series, approximately

$$u = \frac{2^{2n}}{\sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16\cdot 128n^4} + \text{etc.}\right)}\ .$$

Thus the middle term in $(1+1)^{2n}$ is to the sum 2^{2n} of all the terms as

$$1 \text{is to } \sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{1}{16\cdot 128n^4} + \text{etc.}\right);$$

or, if one abbreviates $4n = \nu$, as

$$1 \text{is to } \sqrt{n\pi}\left(1 + \frac{1}{\nu} + \frac{1}{2\nu^2} - \frac{1}{2\nu^3} - \frac{5}{8\nu^4} + \frac{23}{8\nu^5} + \frac{53}{16\nu^6} - \text{etc.}\right).$$

[...]

Second Example

Find the ratio of the middle term of the binomial $(1+1)^{100}$ *to the sum* 2^{100} *of all the terms.*

For this we wish to use the formula we found first,

$$lu = l\frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1\cdot 2\cdot 2n} + \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3n^3} - \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5n^5} + \text{etc.},$$

from which, setting $2n = m$, in order to obtain the power $(1+1)^m$, and after substituting the values of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., one has

$$lu = l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \text{etc.}$$

Since the logarithms here are hyperbolic, one multiplies by

$$k = 0,434294481903251,$$

in order to change to tables, yielding

$$lu = l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17k}{112m^7} - \frac{31k}{36m^9} + \text{etc.},$$

Now since u is the middle coefficient, the ratio sought is $2^m : u$, and

$$l\frac{2^m}{u} = l\sqrt{\frac{1}{2}}m\pi + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{112m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \text{etc.}$$

Now, since the exponent $m = 100$,

$$\frac{k}{m} = 0,0043429448, \quad \frac{k}{m^3} = 0,0000004343, \quad \frac{k}{m^5} = 0,0000000000,$$

yielding

$$\begin{aligned}
\tfrac{k}{4m} &= 0,0010857362 \\
\tfrac{k}{24m^3} &= \underline{0,0000000181} \\
&\ \underline{0,0010857181}\,. \\
\text{Further } l\pi &= 0,4971498726 \\
l\tfrac{1}{2}m &= \underline{1,6989700043} \\
l\tfrac{1}{2}m\pi &= \underline{2,1961198769} \\
l\sqrt{\tfrac{1}{2}m\pi} &= 1,0980599384 \\
\tfrac{k}{4m} - \tfrac{k}{24m^3} + \text{etc.} &= \underline{0,0010857181} \\
&\ 1,0991456565 = l\tfrac{2^{100}}{u}.
\end{aligned}$$

Thus $\frac{2^{100}}{u} = 12,56451$, and the middle term in the expanded power $(1+1)^m$ is to the sum of all the terms 2^{100} as 1 is to $12,56451$.

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