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Translation with notes of Euler's paper

Remarques sur un beau rapport entre les series des puissances tant directes que reciproques

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Translated by
Lucas Willis and Thomas J Osler
Mathematics Department
Rowan University
Glassboro, NJ 08028

Osler@rowan.edu

Introduction to the translation and notes:

This translation is the result of a happy collaboration between student and professor. Lucas Willis is an undergraduate Mathematics Major and a French Minor. Tom Osler has been a mathematics professor for 45 years. Together we struggled to understand this brilliant work.

When translating Euler's words, we tried to imagine how he would have written had he been fluent in modern English and familiar with today's mathematical jargon. Often he used very long sentences, and we frequently converted these to several shorter ones. However, in almost all cases we kept his original notation, even though some is very dated. We thought this added to the charm of the paper. One exception is Euler's use of *lx* for our *log x,* the natural logarithm. We thought *lx* was too confusing..

Euler was very careful in proof reading his work, and we found few typos. When we found an error, we called attention to it in parenthesis and italics in the body of the translation. Other errors are probably ours.

When half the translation was completed, we learned that Professor Robert Stein had made a translation of this paper several years earlier. He generously shared his translation with us, and we gratefully acknowledge his help.

The notes that follow this translation are a collection of material that we accumulated while trying to understand and appreciate Euler's ideas. In these notes we completed some steps that Euler omitted, added some historical remarks, introduced some modern notation and modern mathematical thoughts, especially on the use of divergent series.

Remarks on a beautiful relation between direct as well as reciprocal power series (*E 352*)

By Leonhard Euler

1. The relation, which I intend to develop here, concerns the sums of these

two general infinite series:

$$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \&c.$$

$$\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \&c.$$

The first contains all the positive powers of the natural numbers of a variable m, and the other negative or reciprocal powers of the same natural numbers, of a variable n, while alternating the signs of the terms of both series. My principal goal is to show that though these two series are entirely different, their sums have a very beautiful relationship between them. If we know the sum of one of these two series, we might deduce the sum of the other series. I will show that by knowing the sum of the first series, for a variable m, we can almost always determine the sum of the other series for the variable $n = m + 1$. It seems important to remark that while I only demonstrate this relation for certain special cases, my argument is carried to such a degree of certainty that the reader will conclude it very rigorously shown.

2. For the series of the first type, since the terms become increasingly large, it is quite true that we could not create a correct idea of their sum, if we understand by the sum, a value, that we all the more approach the more we add terms to the series. Thus, when it is said that the sum of this series 1-2+3-4+5-6 etc. is 1/4 , that must appear paradoxical. For by adding 100 terms of this series, we get –50, however, the sum of 101 terms gives +51, which is quite different from 1/4 and becomes still greater when one increases the number of terms. But I have already noticed at a previous time, that it is necessary to give to the word sum a more extended meaning. We understand the sum to be the numerical value, or analytical relationship which is arrived at according to principles of analysis, that generate the same series for which we

seek the sum. After having established this relationship, it is no more doubtful that the sum of this series 1-2+3-4+5 + etc. is 1/4; since it arises from the expansion of the formula $\frac{1}{(1+1)^2}$, whose value is incontestably 1/4 . The idea becomes clearer by considering the general series:

$$1-2x+3x^2-4x^3+5^4-6x^5+\&c.$$

that arises while expanding the expression $\frac{1}{(1+x)^2}$, which this series is indeed equal to after we set $x=1$.

3. It is easy to use the differential calculus to find the sums of these series, and we obtain the following summations:

$$1-x+x^2-x^3+\&c.=\frac{1}{1+x},$$

$$1-2x+3x^2-4x^3+\&c.=\frac{1}{(1+x)^2},$$

$$1-2^2x+3^2x^2-4^2x^3+\&c.=\frac{1-x}{(1+x)^3},$$

$$1-2^3x+3^3x^2-4^3x^3+\&c.=\frac{1-4x+xx}{(1+x)^4},$$

$$1-2^4x+3^4x^2-4^4x^3+\&c.=\frac{1-11x+11xx-x^3}{(1+x)^5},$$

$$1-2^5x+3^5x^2-4^5x^3+\&c.=\frac{1-26x+66xx-26x^3+x^4}{(1+x)^6},$$

$$1-2^6x+3^6x^2-4^6x^3+\&c.=\frac{1-57x+302xx-302x^3+57x^4-x^5}{(1+x)^7},$$

&c.

We obtain our series of the first type by taking $x = 1$, and get the following sums:

$$1-2^0+3^0-4^0+5^0-6^0+\&c.=\frac{1}{2}$$

$$1-2\ \ +3\ \ -4\ \ +5\ \ -6\ \ +\&c.=\frac{1}{4}$$

$$1-2^2+3^2-4^2+5^2-6^2+\&c.=0$$

$$1-2^3+3^3-4^3+5^3-6^3+\&c.=-\frac{2}{16}$$

$$1-2^4+3^4-4^4+5^4-6^4+\&c.=0$$

$$1-2^5+3^5-4^5+5^5-6^5+\&c.=+\frac{16}{64}$$

$$1-2^6+3^6-4^6+5^6-6^6+\&c.=0$$

$$1-2^7+3^7-4^7+5^7-6^7+\&c.=-\frac{272}{256}$$

$$1-2^8+3^8-4^8+5^8-6^8+\&c.=0$$

$$1-2^9+3^9-4^9+5^9-6^9+\&c.=+\frac{7936}{1024}$$

&c.

4. As to the series of the other type, we previously knew only the case n=1, which is

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\&c.$$

whose sum is log 2. First I discovered the sum of the reciprocal series with square powers, and then the sum for all the other even powers. I have shown that the sums of all these series depend on π, the circumference of a circle of diameter 1.

I found the following sums of these series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.=A\pi^2 \qquad A=\frac{1}{6},$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.=B\pi^4 \qquad B=\frac{2}{5}A^2,$$

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c.=C\pi^6 \qquad C=\frac{4}{7}AB,$$

$$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\&c.=D\pi^8 \qquad D=\frac{4}{9}AC+\frac{2}{9}B^2,$$

$$\frac{1}{1^{10}}+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\&c.=E\pi^{10} \qquad E=\frac{4}{11}AD+\frac{4}{11}BC,$$

&c.

from which I calculated sums of our series of the second type with alternating signs

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\&c.=\frac{2\ -1}{2}A\pi^2$$

$$\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\&c.=\frac{2^3-1}{2^3}B\pi^4$$

$$\frac{1}{1^6}-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\&c.=\frac{2^5-1}{2^5}C\pi^6$$

$$\frac{1}{1^8}-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\frac{1}{6^8}+\&c.=\frac{2^7-1}{2^7}D\pi^8$$

$$\frac{1}{1^{10}}-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\frac{1}{6^{10}}+\&c.=\frac{2^9-1}{2^9}E\pi^{10}$$

$$\frac{1}{1^{12}}-\frac{1}{2^{12}}+\frac{1}{3^{12}}-\frac{1}{4^{12}}+\frac{1}{5^{12}}-\frac{1}{6^{12}}+\&c.=\frac{2^{11}-1}{2^{11}}F\pi^{12}$$

&c.

However, in the cases where *n* is an odd number, all my effort to find their sum is a failure up to now. Nevertheless it is certain that they do not depend in a similar way on the powers of the number π. Perhaps the following observations will spread some light here.

5. Since the numbers *A, B, C, D*, etc. are of the highest importance in this subject, I will list them here as far as I have calculated them.

$$A=\frac{2^0\cdot 1}{1\cdot 2\cdot 3},\qquad B=\frac{2^2\cdot 1}{1\cdot 2\cdots 5\cdot 3},$$

$$C=\frac{2^4\cdot 1}{1\cdot 2\cdots 7\cdot 3},\qquad D=\frac{2^6\cdot 3}{1\cdot 2\cdots 9\cdot 5},$$

$$E=\frac{2^8\cdot 5}{1\cdot 2\cdots 11\cdot 3},\qquad F=\frac{2^{10}\cdot 691}{1\cdot 2\cdots 13\cdot 105},$$

$$G=\frac{2^{12}\cdot 35}{1\cdot 2\cdots 15\cdot 1},\qquad H=\frac{2^{14}\cdot 3617}{1\cdot 2\cdots 17\cdot 15},$$

$$I=\frac{2^{16}\cdot 43867}{1\cdot 2\cdots 19\cdot 21},\qquad K=\frac{2^{18}\cdot 1222277}{1\cdot 2\cdots 21\cdot 55},$$

$$L=\frac{2^{20}\cdot 854513}{1\cdot 2\cdots 23\cdot 3},\qquad M=\frac{2^{22}\cdot 1181820455}{1\cdot 2\cdots 25\cdot 273},$$

$$N=\frac{2^{24}\cdot 76977927}{1\cdot 2\cdots 27\cdot 273},\qquad O=\frac{2^{26}\cdot 23749461029}{1\cdot 2\cdots 29\cdot 15},$$

$$P=\frac{2^{28}\cdot 8615841276005}{1\cdot 2\cdots 31\cdot 231},\qquad Q=\frac{2^{30}\cdot 84802531453387}{1\cdot 2\cdots 33\cdot 85},$$

$$R = \frac{2^{32} \cdot 90219075042845}{1 \cdot 2 \cdots 35 \cdot 3}.$$

6. The summation of the series of the first type in the cases, where the variable *m* is an odd number, also involves these same numbers *A, B, C, D* etc. We recall that when this variable is an even number, the sum becomes equal to zero. A method should be used that reveals this beautiful dependence. To achieve this demonstration, it is necessary to use a general method that I have previously published to determine the sums of the series of general terms. Let *X*, be a general function of *x*, and let it be represented by $X = f : x$. Let us consider the infinite series

$$f : x + f : (x+\alpha) + f : (x+2\alpha) + f : (x+3\alpha) + f : (x+4\alpha) + \&c.$$

where the following terms are functions of $x+\alpha, x+2\alpha, x+3\alpha$, etc. Let us call the sum of this series *S*, which is also a function of *x*. If we put $x+\alpha$ in place of *x*, it becomes

$$S + \frac{\alpha\ d\ S}{1dx} + \frac{\alpha^2 dd\ S}{1 \cdot 2\, dx^2} + \frac{\alpha^3 d^3 S}{1 \cdot 2 \cdot 3\, dx^3} + \frac{\alpha^4 d^4 S}{1 \cdot 2 \cdot 3 \cdot 4 dx^4} + \&c.$$

This expression is the sum of the series

$$f : (x+\alpha) + f : (x+2\alpha) + f : (x+3\alpha) + f : (x+4\alpha) + \&c.$$

and is equal to $S - f : x = S - X$, so that

$$-X = \frac{\alpha\ d\ S}{1dx} + \frac{\alpha^2 dd\ S}{1 \cdot 2\, dx^2} + \frac{\alpha^3 d^3 S}{1 \cdot 2 \cdot 3\, dx^3} + \frac{\alpha^4 d^4 S}{1 \cdot 2 \cdot 3 \cdot 4 dx^4} + \&c.$$

However, from this equation I previously derived the formula

$$S = -\frac{1}{\alpha}\int X dx + \frac{1}{2}X - \frac{\alpha\ Ad\ X}{2dx} + \frac{\alpha^3 B d^3 X}{2^3\, dx^3} - \frac{\alpha^5 C d^5 X}{2^5\, dx^5} + \&c.$$

where *A, B, C*, etc are the same numbers which I have just developed. By this means we arrive at the desired sum *S,* using the integral $\int Xdx$, and the derivatives of every order of the function *X.*

7. Now, to obtain the alternating signs, in place of α let us write 2α to get the summation:

$$f:x+f:(x+2\alpha)+f:(x+4\alpha)+\&c.=-\frac{1}{2\alpha}\int Xdx+\frac{1}{2}X$$
$$-\frac{\alpha AdX}{dx}+\frac{\alpha^3 Bd^3X}{dx_3^3}-\frac{\alpha^5 Cd^5X}{dx_3^5}+\&c.$$

and subtracting twice this from the preceding series, we get

$$f:x-f:(x+\alpha)+f:(x+2\alpha)-f:(x+3\alpha)+f:(x+4\alpha)-\&c.$$

$$=\frac{1}{2}X-\frac{(2^2-1)\alpha\ \ Ad\ \ X}{2\ \ dx}+\frac{(2^4-1)\alpha^3 Bd^3X}{2^3dx^3}-\frac{(2^6-1)\alpha^5 Cd^5X}{2^5dx^5}+\&c.$$

where the term, which contains the integral $\int Xdx$, has disappeared. Let us proceed now to our goal by letting $f: x = X = x^m$, and obtain the following sum of the series:

$$x^m-(x+\alpha)^m+(x+2\alpha)^m-(x+3\alpha)^m+(x+4\alpha)^m-\&c.=$$

$$\frac{1}{2}x^m-\frac{(2^2-1)m\alpha\ \ Ax^{m-1}}{2}+\frac{(2^4-1)m(m-1)(m-2)\alpha^3 Bx^{m-3}}{2^3}$$

$$-\frac{(2^6-1)m(m-1)(m-2)(m-3)(m-4)\alpha^5 Cx^{m-5}}{2^5}$$

$$+\frac{(2^8-1)m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)\alpha^7 Dx^{m-7}}{2^7},$$

&c.

which contains only a finite number of terms, when the variable *m* is a positive integer. Therefore, setting $\alpha = 1$, we will have for our series of the first type

$$x^m - (x+1)^m + (x+2)^m - (x+3)^m + (x+4)^m - (x+5)^m + \&c. =$$

$$\frac{1}{2}x^m - \frac{m}{2}\left(2^2 - 1\right)Ax^{m-1} + \frac{m(m-1)(m-2)}{2\cdot\ 2\cdot\ 2}\left(2^4 - 1\right)Bx^{m-3}$$

$$-\frac{m(m-1)(m-2)(m-3)(m-4)}{2\cdot\ 2\cdot\ 2\cdot\ 2\cdot\ 2}\left(2^6 - 1\right)Cx^{m-5}$$

$$+\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)}{2\cdot\ 2\cdot\ 2\cdot\ 2\cdot\ 2\cdot\ 2\cdot\ 2}\left(2^8 - 1\right)Dx^{m-7},$$

&c.

8. Now, we have only to let $x = 1$, to obtain the general sum of all our series of the first type. However it is easier to find the sum by letting $x = 0$, from which we get

$$0^m - 1^m + 2^m - 3^m + 4^m - 5^m + 6^m - 7^m + \&c.$$

which is the negative of the sum we seek. By letting $x = 0$, all the numbers in the sum disappear, except for one, where the power of *x* is zero. This occurs whenever *m* is an odd number, because when it is even, all the members disappear and the sum of the series is reduced to zero. Therefore, taking the negative of these sums, we find the following,

$m = 0$ $\qquad 1 - 1 + 1 - 1 + 1 - \&c. = \frac{1}{2}$

$m = 1$ $\qquad 1 - 2 + 3 - 4 + 5 - 6 + \&c. = +1\frac{\left(2^2 - 1\right)}{2}A,$

$m = 2$ $\qquad 1 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \&c. = 0,$

$m = 3$ $\qquad 1 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + \&c. = -1\cdot 2\cdot 3\frac{\left(2^4 - 1\right)}{2^3}B,$

$m=4 \qquad 1-2^4+3^4-4^4+5^4-6^4+\&c.=0$,

$m=5 \qquad 1-2^5+3^5-4^5+5^5-6^5+\&c.=+1\cdot 2\cdot\cdot 5\cdot\dfrac{\left(2^6-1\right)}{2^5}C$,

$m=6 \qquad 1-2^6+3^6-4^6+5^6-6^6+\&c.=0$,

$m=7 \qquad 1-2^7+3^7-4^7+5^7-6^7+\&c.=-1\cdot 2\cdot\cdot 7\cdot\dfrac{\left(2^8-1\right)}{2^7}D$,

$m=8 \qquad 1-2^8+3^8-4^8+5^8-6^8+\&c.=0$,

$m=9 \qquad 1-2^9+3^9-4^9+5^9-6^9+\&c.=+1\cdot 2\cdot\cdot 9\cdot\dfrac{\left(2^{10}-1\right)}{2^9}E$,

$m=10 \qquad 1-2^{10}+3^{10}-4^{10}+5^{10}-6^{10}+\&c.=0$,

&c.

When these sums are calculated, it is found that they are the same values that I listed above, but now we see their connection with the values *A, B, C,* etc.

9. We divide each one of these series of the first type by that series of the second type, which contains the same number *A, B, C, D,* etc to obtain the following equations.

$$\frac{1-2\;\;+3\;\;-4\;\;+5\;\;-6\;\;+\&c.}{1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\&c.}=+\frac{1\left(2^2-1\right)}{\left(2\;\;-1\right)\pi^2},$$

$$\frac{1-2^2+3^2-4^2+5^2-6^2+\&c.}{1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{6^3}+\&c.}=0,$$

$$\frac{1-2^3+3^3-4^3+5^3-6^3+\&c.}{1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\&c.}=-\frac{1\cdot 2\cdot 3\cdot\left(2^4-1\right)}{\left(2^3-1\right)\pi^4},$$

$$\frac{1-2^4+3^4-4^4+5^4-6^4+\&c.}{1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\frac{1}{6^5}+\&c.}=0,$$

$$\frac{1-2^5+3^5-4^5+5^5-6^5+\&c.}{1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\&c.}=+\frac{1\cdot 2\cdot\cdot 5\cdot\left(2^6-1\right)}{\left(2^5-1\right)\pi^6},$$

$$\frac{1-2^6+3^6-4^6+5^6-6^6+\&c.}{1-\frac{1}{2^7}+\frac{1}{3^7}-\frac{1}{4^7}+\frac{1}{5^7}-\frac{1}{6^7}+\&c.}=0,$$

$$\frac{1-2^7+3^7-4^7+5^7-6^7+\&c.}{1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\frac{1}{6^8}+\&c.}=+\frac{1\cdot 2\cdot\cdot 7\cdot\left(2^8-1\right)}{\left(2^7-1\right)\pi^8},$$

$$\frac{1-2^8+3^8-4^8+5^8-6^8+\&c.}{1-\frac{1}{2^9}+\frac{1}{3^9}-\frac{1}{4^9}+\frac{1}{5^9}-\frac{1}{6^9}+\&c.}=0$$

$$\frac{1-2^9+3^9-4^9+5^9-6^9+\&c.}{1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\frac{1}{6^{10}}+\&c.}=-\frac{1\cdot 2\cdot\cdot 9\cdot\left(2^{10}-1\right)}{\left(2^9-1\right)\pi^{10}},$$

&c.

(*Misprint in the 3rd equation above: the term* $\frac{1}{6^4}$ *is mistakenly printed as* $\frac{1}{6^5}$.)

But the equation which precedes these, is

$$\frac{1-1+1-1+1-1+\&c.}{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\&c.}=\frac{1}{2\log 2}$$

whose connection with the following ones is entirely hidden.

10. Consideration of these equations leads me to this general formula:

$$\frac{1-2^{n-1}+3^{n-1}-4^{n-1}+5^{n-1}-6^{n-1}+\&c.}{1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\&c.}=+N\cdot\frac{1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)\left(2^n-1\right)}{\left(2^{n-1}-1\right)\pi^n},$$

where we must now proceed to precisely determine the coefficient N of the variable *n*. To achieve this evaluation, let us consider the values of this coefficient *N*, which correspond to each variable *n*, that I have just examined:

n	2,	3,	4,	5,	6,	7,	8,	9,	10,	*etc.*
N	+1,	0,	−1,	0,	+1,	0,	−1,	0,	+1	*etc.*

Since whenever *n* is an odd number, the letter *N* must disappear, and for the case $n = 4i + 2$, it is necessary that it is *N* = 1; but for the case $n = 4i$, it becomes $N = -1$, it is obvious that we can satisfy these conditions by taking $N = -\cos(\pi n/2)$: For this reason I venture to propose the following conjecture, that for any variable *n*, the following equation is always valid:

$$\frac{1-2^{n-1}+3^{n-1}-4^{n-1}+5^{n-1}-6^{n-1}+\&c.}{1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\&c.} = \frac{-1\cdot 2\cdot 3\cdots(n-1)\left(2^n-1\right)}{\left(2^{n-1}-1\right)\pi^n}\cos\frac{n\pi}{2}.$$

This conjecture will undoubtedly stand out as bold, but since it has been shown to be true for the case where *n* is a positive integer larger than one, I shall next prove this relation for the case *n* = 1, and then for *n* = 0. After that I will show, that if this conjecture is proved for the cases where *n* is a positive integer, it will also be true when *n* is a negative integer. Finally I will demonstrate some cases where we give to *n* a fractional value.

11. First let *n* = 1, and get the expression

$$\frac{1-1+1-1+1-1+\&c.}{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\&c.}$$

whose value is $1/(2\log 2)$. However for this case our conjectured relation contains the expressions $1\cdot 2\cdot 3\cdots(n-1)=1$ and $\pi^n = \pi$, while the two other expressions $\cos(n\pi/2)$

and $2^{n-1}-1$, are both zero, with one dividing the other. This is why I write the expression, that our conjecture gives in this case as:

$$-\frac{1}{\pi}\cdot\frac{\cos\frac{n\pi}{2}}{2^{n-1}-1},$$

where it is a question of determining the value of the fraction $\frac{\cos\frac{n\pi}{2}}{2^{n-1}-1}$, where the numerator and the denominator are both zero. Now let us treat the letter *n* as a variable, and since the differential of the numerator is $-\frac{\pi\,dn}{2}\sin\left(\frac{n\pi}{2}\right)$, and that of the denominator is $2^{n-1}dn\log 2$, our fraction for this case will be the same as $-\frac{\frac{\pi}{2}\sin\left(\frac{n\pi}{2}\right)}{2^{n-1}\log 2}$. Letting $n=1$ this is reduced to $-\frac{\pi}{2\log 2}$, so that the value that we seek will be

$$-\frac{1}{\pi}\frac{\cos(n\pi/2)}{2^{n-1}-1}=\frac{1}{2\log 2}.$$

Thus our conjecture also holds for the case *n* = 1, which initially appeared to entirely deviate from the rule of the prior cases. This is already a type of proof for the truth of this conjecture, for it seems impossible that a false supposition could support this test. We may already look at our conjecture as very firmly established. However, I am going to bring even more convincing evidence.

12. Let *n*= 0, and now we must consider the expression

$$\frac{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\&c.}{1-1+1-1+1-1+\&c.}$$

whose value is obviously equal to 2log2. However from our conjecture, we have the expressions $\cos(n\pi/2)=1$, and $\pi^n=1$, as well as $2\cdot1\cdot2\cdot3\cdots(n-1)(2^n-1)$. The factor $1\cdot2\cdot3\cdots(n-1)$ is infinite, and the other expression 2^n-1 is zero, from which we see that our conjecture is not yet contradicted in this case. But, to proceed to a proof, I notice that

$$1\cdot2\cdot3\cdots\cdot(n-1)=\frac{1}{n}1\cdot2\cdot3\cdots\cdot n$$

and in the case $n=0$, we have $1\cdot2\cdot3\cdots\cdot n=1$. Therefore in this same case $1\cdot2\cdot3\cdots\cdot(n-1)=\frac{1}{n}$, and the value from our conjecture $=\frac{2\left(2^n-1\right)}{n}$, where since the numerator and denominator disappear when putting $n=0$, we have only to substitute for them their differentials. Thus we have another fraction $\frac{2\cdot2^n\,dn\log2}{dn}=2\cdot2^n\log2$, equivalent to the one for the case $n=0$. Now this one gives us the same value $2\log2$ that the nature of the series demands. Here is therefore a new verification, which being joined to the preceding one will be able to give us of a more complete demonstration of our conjecture. Nevertheless we have not given a direct demonstration that contains at once all the possible cases.

13. Our conjecture being verified only for all the case where *n* is a positive integer, I am going to prove now that it is equally true when we take for *n* any negative integer. In these cases the value of the expression $1\cdot2\cdot3\cdots(n\text{-}1)$ is infinity, and this seems to invalidate the conjecture that I have in mind; however, an evaluation that I made previously will overcome this obstacle. Take this notation, $[\lambda]$, to represent the

product: $1\cdot 2\cdot 3\cdots\lambda$. I have shown previously that it is always true that

$[\lambda]\cdot[-\lambda]=\dfrac{\lambda\pi}{\sin\lambda\pi}$. Thus letting $n-1=-m$ or $n=-m+1$, we get the expression

$$\frac{1-2^{-m}+3^{-m}-4^{-m}+5^{-m}-6^{-m}+\&c.}{1-2^{m-1}+3^{m-1}-4^{m-1}+5^{m-1}-6^{m-1}+\&c.}=$$
$$\frac{-1\cdot 2\cdot 3\cdots(-m)\left(2^{-m+1}-1\right)}{\left(2^{-m}-1\right)\pi^{-m+1}}\cos\frac{(1-m)\pi}{2}$$

where since $1\cdot 2\cdot 3\cdots(-m)=[-m]$ and $[m]\cdot[-m]=\dfrac{m\pi}{\sin m\pi}$, we will have

$$1\cdot 2\cdot 3\cdots(-m)=\frac{m\pi}{1\cdot 2\cdot 3\cdots m\sin m\pi}=\frac{\pi}{1\cdot 2\cdot 3\cdots(m-1)\sin m\pi}.$$

Then since $\cos((1-m)\pi/2)=\sin(m\pi/2)$, the expression from our conjecture takes the following form upon making these substitutions:

$$-\frac{2\left(2^{m-1}-1\right)\pi^{m}}{\left(2^{m}-1\right)1\cdot 2\cdot 3\cdots\cdot(m-1)\sin m\pi}\sin\frac{m\pi}{2}=$$

$$-\frac{\left(2^{m-1}-1\right)\pi^{m}}{1\cdot 2\cdot 3\cdots\cdot(m-1)\left(2^{m}-1\right)\cos\frac{m\pi}{2}},$$

where we have used $\sin m\pi=2\sin(m\pi/2)\cos(m\pi/2)$. Now, we have only to invert the equation found by putting on top the denominator and on bottom the numerator, and we will obtain the equation:

$$\frac{1-2^{m-1}+3^{m-1}-4^{m-1}+5^{m-1}-6^{m-1}+etc.}{1-2^{-m}+3^{-m}-4^{-m}+5^{-m}-6^{-m}+etc.}=$$
$$-\frac{1\cdot 2\cdot 3\cdots(m-1)\left(2^{m}-1\right)}{\left(2^{m-1}-1\right)\pi^{m}}\cos\frac{m\pi}{2}$$

which is the same one that was conjectured. It is seen clearly that if the conjectured expression is correct for the case where n is a positive number, it will be true also when n is a negative number, because of $m=-n+1$.

14. A remarkable case is found by setting n = 1/2 , which leads to this fraction

$$\frac{1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}+\&c.}{1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}+\&c.}$$

whose numerator and denominator being equal, gives the value = 1: We must find the value of the expression, which the conjecture suggests it is equal to:

$$-\frac{1\cdot 2\cdot 3\cdots\left(-\frac{1}{2}\right)\left(\sqrt{2}-1\right)}{\left(\frac{1}{\sqrt{2}}-1\right)\sqrt{\pi}}\cos\frac{\pi}{4}=+\frac{\left[-\frac{1}{2}\right]\sqrt{2}}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{2}}=\frac{\left[-\frac{1}{2}\right]}{\sqrt{\pi}}$$

However, I have shown before, by examining the factorial progression $1;1\cdot 2$; $1\cdot 2\cdot 3$, $1\cdot 2\cdot 3\cdot 4,\cdots$; whose general term is $1\cdot 2\cdot 3\cdots n=[n]$, that setting n = 1/2 , we have $[1/2]=\sqrt{\pi}/2$. Because $[1/2]=(1/2)[-1/2]$, it is evident that $[-1/2]=\sqrt{\pi}/2$, (*this should be* $=\sqrt{\pi}$), which indeed leaves our expression equal to 1. There should not be any further doubt about our conjecture, having verified it not only for all the cases where the variable n is an integer, be it n is positive, or n is negative, but also for the case n = 1/2 . For the other cases involving fractional numbers, that we would like to use instead of n, we cannot claim a particular proof, considering that no one has yet discovered a correct method to determine the sum of a series $1-2^n+3^n-4^n+5^n\cdots$ when the variable n is a fraction. In these cases it is necessary to be satisfied with numerical approximations: however, we will see that our conjecture remains true.

15. To carry out a test, let $n=3/2$. Because

$$1\cdot 2\cdot 3\cdots(n-1)=\left[\frac{1}{2}\right]=\frac{1}{2}\sqrt{\pi}\text{ , and }\cos\frac{3\pi}{4}=-\frac{1}{\sqrt{2}},$$

this fraction

$$\frac{1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\sqrt{6}+\&c.}{1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}-\frac{1}{6\sqrt{6}}+\&c.}$$

must be equal to this quantity

$$\frac{2\sqrt{2}-1}{2\left(2-\sqrt{2}\right)\pi}=\frac{3+\sqrt{2}}{2\pi\sqrt{2}}=0,4967738\,.$$

But on calculating the first 9 terms of the series in the numerator, we get

$$1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\sqrt{6}+\sqrt{7}-\sqrt{8}+\sqrt{9}=1,9217396662\,,$$

from which it was necessary ad infinitum to cut off the sum of the following terms

$$\sqrt{10}-\sqrt{11}+\sqrt{12}-\sqrt{13}+\sqrt{14}-\&c.$$

From section 7 this is

$$\frac{1}{2}\sqrt{10}-\frac{1\left(2^2-1\right)}{4}\cdot\frac{A}{\sqrt{10}}+\frac{1\cdot 1\cdot 3\left(2^4-1\right)}{4^3}\cdot\frac{B}{10^2\sqrt{10}}-$$
$$\frac{1\cdot 1\cdot 3\cdot 5\cdot 7\left(2^6-1\right)}{4^5}\cdot\frac{C}{10^3\sqrt{10}}+\frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\left(2^8-1\right)}{4^7}\cdot\frac{D}{10^4\sqrt{10}}-\&c.$$

$$=\frac{\sqrt{10}}{2}\left(1-\frac{1\cdot 3}{2}\cdot\frac{A}{\sqrt{10}}+\frac{1\cdot 1\cdot 3\cdot 15}{2^5}\cdot\frac{B}{10^3}-\frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 63}{2^9}\cdot\frac{C}{10^5}+\frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 255}{2^{13}}\cdot\frac{D}{10^7}-\&c.\right)$$

and after substituting the values

$A=\frac{1}{6}$, $B=\frac{1}{90}$, $C=\frac{1}{945}$, $D=\frac{1}{9450}$, $E=\frac{1}{93555}$, &c., we calculate

=0,48750774577·√10, which is about =1,541610, (*should be 1.541634853*) and

making the numerator series : *1-√2+√3-√4+√5-√6+...*= 0,380129. (*should be 0.380104812*) Now, for the lower series, the first 9 terms gives 0,7821470744, (*should be 0.782135824*) from which it is necessary to cut off the sum from all of the following, which is

$$\frac{1}{20\sqrt{10}}\left(1+\frac{3\cdot 3}{2}\cdot\frac{A}{10}-\frac{3\cdot 5\cdot 7\cdot 15}{2^5}\cdot\frac{B}{10^3}-\frac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 63}{2^9}\cdot\frac{C}{10^5}-\&c.\right)$$

and is about = 0,01698880, and thus the sum of this infinite series will be = 0,765158 (*should be 0.765147024*) Now let us see if the first series divided by this one, which is the fraction $\frac{0,380129}{0,765158}$ (*should be 0.380104812 / 0.765147024 = 0.496773561*) is equal to the value 0,4967738. The difference is so small, being only two hundred-thousandths of the unit, (*should be 2 ten millionths*) that one could not doubt in the least that this matter is true.

16. Since our conjecture has been demonstrated to the highest degree of certainty, so that there remains no more doubt of its validity when *n* is a fraction, we list results for the case where *n* is a fraction of the form $(2i+1)/2$:

$$\frac{1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\&c.}{1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\&c.}=+\frac{\left(2\sqrt{2}-1\right)}{2\left(2-\sqrt{2}\right)\pi}$$

$$\frac{1-2\sqrt{2}+3\sqrt{3}-4\sqrt{4}+\&c.}{1-\frac{1}{2^2\sqrt{2}}+\frac{1}{3^2\sqrt{3}}-\frac{1}{4^2\sqrt{4}}+\&c.}=+\frac{1\cdot 3\left(4\sqrt{2}-1\right)}{2^2\left(4-\sqrt{2}\right)\pi^2}$$

$$\frac{1-2^2\sqrt{2}+3^2\sqrt{3}-4^2\sqrt{4}+\&c.}{1-\frac{1}{2^3\sqrt{2}}+\frac{1}{3^3\sqrt{3}}-\frac{1}{4^3\sqrt{4}}+\&c.}=-\frac{1\cdot 3\cdot 5\left(8\sqrt{2}-1\right)}{2^3\left(8-\sqrt{2}\right)\pi^3}$$

$$\frac{1-2^3\sqrt{2}+3^3\sqrt{3}-4^3\sqrt{4}+\&c.}{1-\frac{1}{2^4\sqrt{2}}+\frac{1}{3^4\sqrt{3}}-\frac{1}{4^4\sqrt{4}}+\&c.}=-\frac{1\cdot 3\cdot 5\cdot 7\left(16\sqrt{2}-1\right)}{2^4\left(16-\sqrt{2}\right)\pi^4}$$

$$\frac{1-2^4\sqrt{2}+3^4\sqrt{3}-4^4\sqrt{4}+\&c.}{1-\frac{1}{2^5\sqrt{2}}+\frac{1}{3^5\sqrt{3}}-\frac{1}{4^5\sqrt{4}}+\&c.}=+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\left(32\sqrt{2}-1\right)}{2^5\left(32-\sqrt{2}\right)\pi^5}$$

$$\frac{1-2^5\sqrt{2}+3^5\sqrt{3}-4^5\sqrt{4}+\&c.}{1-\frac{1}{2^6\sqrt{2}}+\frac{1}{3^6\sqrt{3}}-\frac{1}{4^6\sqrt{4}}+\&c.}=+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\left(64\sqrt{2}-1\right)}{2^6\left(64-\sqrt{2}\right)\pi^6}$$

$$\frac{1-2^6\sqrt{2}+3^6\sqrt{3}-4^6\sqrt{4}+\&c.}{1-\frac{1}{2^7\sqrt{2}}+\frac{1}{3^7\sqrt{3}}-\frac{1}{4^7\sqrt{4}}+\&c.}=-\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\left(128\sqrt{2}-1\right)}{2^7\left(128-\sqrt{2}\right)\pi^7}$$

It should be noted that $\frac{2^\lambda\sqrt{2}-1}{2^\lambda-\sqrt{2}}$, can be reduced to $\frac{\left(2^{2\lambda}-1\right)\sqrt{2}+2^\lambda}{2^{2\lambda}-2}$.

Therefore, with each pair of these series, as soon as we have found the sum of one, we will find from it the sum of the other in a relation involving the number π.

17. Regarding the reciprocal series of the powers

$$1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\&c.$$

I have already observed, that their sums have been found only when *n* is an even integer, and that for the case where *n* is an odd integer, all of my efforts have been completely useless. Now having related the sum of these reciprocal series to that of the direct series, and since we in general know the sum of

$1-2^{n-1}+3^{n-1}-5^{n-1}+6^{n-1}-\&c.$, we could expect to find some way to achieve our goal, but it is unfortunate that whenever *n* is an odd number, the sum of this direct series is zero. Thus we could not conclude anything, because letting $n=2\lambda+1$, by our conjecture we have:

$$1-\frac{1}{2^{2\lambda+1}}+\frac{1}{3^{2\lambda+1}}-\frac{1}{4^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}-\&c.=$$
$$-\frac{\left(2^{2\lambda}-1\right)\pi^{2\lambda+1}}{1\cdot 2\cdot 3\cdots 2\lambda\left(2^{2\lambda+1}-1\right)}\frac{1-2^{2\lambda}+3^{2\lambda}-4^{2\lambda}+5^{2\lambda}-\&c.}{\cos\frac{2\lambda+1}{2}\pi}$$

However, in this last expression, the value of both the numerator

$$1-2^{2\lambda}+3^{2\lambda}-4^{2\lambda}+5^{2\lambda}-\&c.$$

and the denominator $\cos\frac{2\lambda+1}{2}\pi=-\sin\lambda\pi$ are zero, when λ is an integer. It is true that we can easily discover the value of such a fraction, by substituting instead of the numerator and denominator their differentials, however this technique is not successful as I am going to demonstrate.

18. To show this, we find that the differential of the numerator is

$$2d\lambda\left(1^{2\lambda}\log 1-2^{2\lambda}\log 2+3^{2\lambda}\log 3-4^{2\lambda}\log 4+\&c.\right)$$

and that of the denominator is $-\pi\, d\lambda\cos\lambda\pi$. We get for our case the sum expressed in the form

$$1-\frac{1}{2^{2\lambda+1}}+\frac{1}{3^{2\lambda+1}}-\frac{1}{4^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}-\&c.=$$
$$\frac{2\left(2^{2\lambda}-1\right)\pi^{2\lambda}}{1\cdot 2\cdot 3\cdots 2\lambda\left(2^{2\lambda+1}-1\right)\cos\lambda\pi}(1^{2\lambda}\log 1-2^{2\lambda}\log 2+3^{2\lambda}\log 3-4^{2\lambda}\log 4+\&c.)$$

Upon substituting for λ the numbers 1, 2, 3,…, we obtain the following summations:

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\&c.=-\frac{2\cdot 3\cdot \pi^2(1\ \log 1-2^2\log 2+3^2\log 3-4^2\log 4+\&c.)}{1\cdot 2\cdot 7}$$

$$1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\&c.=+\frac{2\cdot 15\cdot \pi^4(1\ \log 1-2^4\log 2+3^4\log 3-4^4\log 4+\&c.)}{1\cdot 2\cdot 3\cdot 4\cdot 31}$$

$$1-\frac{1}{2^7}+\frac{1}{3^7}-\frac{1}{4^7}+\&c.=-\frac{2\cdot 63\cdot \pi^6(1\ \log 1-2^6\log 2+3^6\log 3-4^6\log 4+\&c.)}{1\cdot 2\cdot 3\cdots 6\cdot 127}$$

$$1-\frac{1}{2^9}+\frac{1}{3^9}-\frac{1}{4^9}+\&c.=+\frac{2\cdot 255\cdot \pi^8(1\ \log 1-2^8\log 2+3^8\log 3-4^8\log 4+\&c.)}{1\cdot 2\cdot 3\cdots 8\cdot 511}$$

$$1-\frac{1}{2^{11}}+\frac{1}{3^{11}}-\frac{1}{4^{11}}+\&c.=-\frac{2\cdot 1023\cdot \pi^{10}(1\ \log 1-2^{10}\log 2+3^{10}\log 3-4^{10}\log 4+\&c.)}{1\cdot 2\cdot 3\cdots 10\cdot 2047}$$

Thus it is necessary that we find the sums of the series of the form

$$1^{2\lambda}\log 1-2^{2\lambda}\log 2+3^{2\lambda}\log 3-4^{2\lambda}\log 4+\&c.$$

But this evaluation is perhaps more difficult than that which we have in mind, and I do not foresee any method that can lead us to the desired goal.

19. These equations become a bit simpler upon considering that the series $1+\frac{1}{3^m}+\frac{1}{5^m}+\frac{1}{7^m}+\frac{1}{9^m}+\&c.$ is equal to this one

$$\frac{2^m-1}{2(2^{m-1}-1)}\left(1-\frac{1}{2^m}+\frac{1}{3^m}-\frac{1}{4^m}+\frac{1}{5^m}-\&c.\right).$$

Using the previous methods we find the general sum

$$1+\frac{1}{3^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}+\frac{1}{7^{2\lambda+1}}+\frac{1}{9^{2\lambda+1}}+\&c.=$$
$$-\frac{\pi^{2\lambda}}{1\cdot 2\cdot 3\cdots 2\lambda\cos\lambda\pi}(2^{2\lambda}\log 2-3^{2\lambda}\log 3+4^{2\lambda}\log 4-5^{2\lambda}\log 5+\&c.)$$

and then list the particular cases:

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\&c.=+\frac{\pi^2(2^2\log 2-3^2\log 3+4^2\log 4-\&c.)}{1\cdot 2}$$

$$1+\frac{1}{3^5}+\frac{1}{5^5}+\frac{1}{7^5}+\&c.=-\frac{\pi^4(2^4\log 2-3^4\log 3+4^4\log 4-\&c.)}{1\cdot 2\cdot 3\cdot 4}$$

$$1+\frac{1}{3^7}+\frac{1}{5^7}+\frac{1}{7^7}+\&c.=+\frac{\pi^6(2^6\log 2-3^6\log 3+4^6\log 4-\&c.)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$

$$1+\frac{1}{3^9}+\frac{1}{5^9}+\frac{1}{7^9}+\&c.=-\frac{\pi^8(2^8\log 2-3^8\log 3+4^8\log 4-\&c.)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}$$

&c.

(*Misprint in the 2nd equation above: the term* $\frac{1}{3^5}$ *appears as* $\frac{1}{3^4}$.)

However, here it should be noticed that the general sum in these two previous paragraphs is true only when the variable λ is a positive integer, since it is based on the condition that the sum of the series

$$1-2^{2\lambda}+3^{2\lambda}-4^{2\lambda}+\&c.$$

is zero. This sum is not zero anymore in the case $\lambda=0$, therefore we can use for λ only the numbers 1, 2, 3, 4,… .I note that the series $\log 2-\log 3+\log 4-\log 5+etc.$ has the sum $=\frac{1}{2}\log\frac{\pi}{2}$, which gives us reason to hope for success in finding the sum of the series that lead us here.

20. In the same way, we can compare the sums of these two infinite series $1-3^{n-1}+5^{n-1}-7^{n-1}+\&c.$ and $1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\&c.$ and obtain the similar conjecture

$$\frac{1-3^{n-1}+5^{n-1}-7^{n-1}+\&c.}{1-3^{-n}+5^{-n}-7^{-n}+\&c.}=\frac{1\cdot 2\cdot 3\cdots(n-1)2^n}{\pi^n}\sin\frac{n\pi}{2}.$$

Whenever n is a positive even integer, the numerator sum of the series disappears, and in these cases, also, the sine of the angle becomes zero. Therefore, letting $n=2\lambda$, we have:

$$1-\frac{1}{3^{2\lambda}}+\frac{1}{5^{2\lambda}}-\frac{1}{7^{2\lambda}}+\&c.=-\frac{\pi^{2\lambda-1}(3^{2\lambda-1}\log 3-5^{2\lambda-1}\log 5+7^{2\lambda-1}\log 7-\&c.)}{1\cdot 2\cdot 3\cdots(2\lambda-1)2^{2\lambda-1}\cos\lambda\pi}$$

Letting *n* be a positive integer we calculate the following summations:

$$1-\frac{1}{3^{2}}+\frac{1}{5^{2}}-\frac{1}{7^{2}}+\&c.=+\frac{\pi\ \ (3\ \ \log 3-5\ \ \log 5+7\ \ \log 7-\&c.)}{1\cdot 2^{2}}$$

$$1-\frac{1}{3^{4}}+\frac{1}{5^{4}}-\frac{1}{7^{4}}+\&c.=-\frac{\pi^{3}(3^{3}\log 3-5^{3}\log 5+7^{3}\log 7-\&c.)}{1\cdot 2\cdot 3\cdot 2^{3}}$$

$$1-\frac{1}{3^{6}}+\frac{1}{5^{6}}-\frac{1}{7^{6}}+\&c.=+\frac{\pi^{5}(3^{5}\log 3-5^{5}\log 5+7^{5}\log 7-\&c.)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 2^{5}}$$

$$1-\frac{1}{3^{8}}+\frac{1}{5^{8}}-\frac{1}{7^{8}}+\&c.=-\frac{\pi^{7}(3^{7}\log 3-5^{7}\log 5+7^{7}\log 7-\&c.)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 2^{7}}$$

$$1-\frac{1}{3^{10}}+\frac{1}{5^{10}}-\frac{1}{7^{10}}+\&c.=+\frac{\pi^{9}(3^{9}\log 3-5^{9}\log 5+7^{9}\log 7-\&c.)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 2^{9}}$$

This last conjecture contains an expression simpler than the preceding one, therefore there is hope that further work will bring success. Finding a demonstration of it will not fail to spread much light on a number of other problems of this nature.

Translator's Notes

These notes are keyed to the 20 sections of Euler's paper.

Section 1. The main interest in this paper, for modern readers, is that Euler finds a relation equivalent to the functional equation for the zeta function. The zeta function defined by the series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^{s}},\quad \mathrm{Re}(s)>1,$$

is one of the most important special functions in mathematics. The functional equation for the zeta function is

(1.1) $\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$.

Instead of using $\zeta(s)$, Euler uses the alternating series [1, p. 807]

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},$$

which is defined for the wider region $\mathrm{Re}(s) > 0$. The function $\eta(s)$ is one simple step removed from $\zeta(s)$ as shown by the relation

$$\eta(s) = \left(1-2^{1-s}\right)\zeta(s) \quad .$$

The functional equation now becomes

(1.2) $\left(2^{s-1}-1\right)\eta(1-s) = -\left(2^s-1\right)\pi^{-s} \cos\left(\frac{\pi s}{2}\right)\Gamma(s)\eta(s)$.

This is easily manipulated into Euler's relation when $s = n$, a natural number which he writes as

(1.3) $$\frac{1-2^{n-1}+3^{n-1}-4^{n-1}+5^{n-1}-6^{n-1}+\cdots}{1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\cdots} = \frac{-(n-1)!\left(2^n-1\right)}{\left(2^{n-1}-1\right)\pi^n}\cos\frac{n\pi}{2}.$$

When *n* is a natural number, the series in the numerator diverges, and the modern reader is severely troubled. Euler however, the grand master of series manipulation is undaunted. One of the most interesting features of this paper is Euler's exciting evaluation of divergent series. Throughout his paper, Euler refers to the "direct series"

(1.4) $$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \cdots,$$

and the "reciprocal series"

(1.5) $$\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \cdots,$$

Euler had received much acclaim for finding exact closed form values for the series $\zeta(s)$ for *s* a positive even integer (see section 4 aand 5). So why did he choose to write about the alternating series $\eta(s)$ rather than $\zeta(s)$? Because the alternating series can be summed by his methods, (see section 15), even for values of *s* where the series diverges. This is not true of $\zeta(s)$. Why was he interested in a functional equation?

Probably because he was trying to sum $\zeta(s)$ for s an odd positive integer, an attractive problem that has never been solved.

At the end of his paper, Euler considers another function of interest

$$L(s)=\frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\cdots=\frac{\zeta(s)+\eta(s)}{2},$$

valid for $\mathrm{Re}(s)>0$. Euler finds the functional equation

$$L(1-s)=2^s\pi^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s)L(s)\,. \tag{1.5}$$

Hardy [4, p. 23] writes "*These results have usually been attributed to Riemann, Malmsten, and Schlomilch. It was comparatively recently that it was observed, first by Cahen and then by Landau, that both (1.2), which is equivalent to (1.1), and (1.6) stand in a paper of Euler written in 1749, over 100 years before Riemann*".

Helpful summaries of Euler's ideas from this paper are in Ayoub's article [2] and Hardy's classic book [4, pp. 23-26].

Section 2. Euler explains his idea of the sum of a divergent series. He uses the example of the series 1-2+3-4+5-6 etc., which arises by first expanding

$$\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+5^4-6x^5+\cdots,$$

then setting $x = 1$. We conclude that $1-2+3-4+5-6+etc.=1/4$. In modern terms this is the Abel summation of the series which Hardy [4, p. 7] describes essentially as follows:

Definition of Abel sum. If $\sum_{n=0}^{\infty}a_nx^n=f(x)$ is convergent for $|x|<1$, and

$$\lim_{x\to 1-}f(x)=s,$$

then we call s the Abel sum of $\sum_{n=0}^{\infty}a_n$.

Few modern mathematicians would try using divergent series as a tool to discover mathematical truth. As early as 1826 Niels Heinrik Abel wrote: *The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes. ….*

On the other hand, Hardy writes: *The definitions of convergence and divergence are now commonplaces of elementary analysis. The ideas were familiar to mathematicians before Newton and Leibniz* (*indeed to Archimedes*)*; and all the great mathematicians of the seventeenth and eighteenth centuries, however recklessly they may seem to have manipulated series, knew well enough whether the series which they used converged.* The great electrical engineer Oliver Heaviside (1850–1925) wrote: *The series is divergent; therefore we may be able to do something with it.*

It is the surprising success of Euler's masterful manipulations with divergent series that gives this paper its distinctive flavor and intense interest.

<u>Section 3.</u> Euler lists seven special cases (n = 0 to 6) of the following closed form summation of power series:

$$1-2^n x+3^n x^2-4^n x^3+\cdots=\frac{1-{}_nc_1x+{}_nc_2x^2-{}_nc_3x^3+\cdots+(-1)^n{}_nc_nx^n}{(1+x)^{n+1}}.$$

Euler skips the derivation, so we outline ours here. We start with the case n = 0 which is the geometric series,

$$1-x+x^2-x^3+etc.=\frac{1}{1+x}.$$

To derive the remaining relations, simply multiply the last one by x and differentiate. For example, if we have found

$$(3.1)\qquad 1-2^n x+3^n x^2-4^n x^3+\cdots=\frac{1-{}_nc_1x+{}_nc_2x^2-{}_nc_3x^3+\cdots+(-1)^n{}_nc_nx^n}{(1+x)^{n+1}},$$

then multiplying by x we get

$$x-2^n x^2+3^n x^3-4^n x^4+\cdots=\frac{x-{}_nc_1x^2+{}_nc_2x^3-{}_nc_3x^4+\cdots+(-1)^n{}_nc_nx^{n+1}}{(1+x)^{n+1}},$$

and differentiating we have the next relation

$$1-2^{n+1} x+3^{n+1} x^2-4^{n+1} x^3+\cdots=\frac{1-{}_{n+1}c_1x+{}_{n+1}c_2x^2-{}_{n+1}c_3x^3+\cdots+(-1)^{n+1}{}_{n+1}c_{n+1}x^{n+1}}{(1+x)^{n+2}}.$$

A simple examination of the coefficients ${}_nc_r$ in Euler's list reveals the recurrence relation

$${}_{n+1}c_r=(r+1)\,{}_nc_r+(n-r+1)\,{}_nc_{r-1}$$

in action. We also note that ${}_nc_r={}_nc_{n-r}$.

Now let $x\to 1$ in (3.1) and obtain the Abel summation of the divergent series

(3.2) $$1-2^n+3^n-4^n+\cdots=\frac{1-{}_nc_1+{}_nc_2-{}_nc_3+\cdots+(-1)^n{}_nc_n}{2^{n+1}}.$$

<u>Sections 4 and 5.</u> In previous work, Euler found closed form expressions for $\zeta(z)$ when z is an even natural number. He showed that

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},$$

$$\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},$$

$$\zeta(6)=\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945},$$

and, today we write in general

(4.1) $$\zeta(2p)=\sum_{n=1}^{\infty}\frac{1}{n^{2p}}=\frac{(-1)^{p+1}2^{2p-1}B_{2p}}{(2p)!}\pi^{2p}.$$

(See Knopp [5, page 237].) Using $\eta(s)=\left(1-2^{1-s}\right)\zeta(s)$ we have

(4.2) $$\eta(2p)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2p}}=\frac{(-1)^{p+1}\left(2^{2p-1}-1\right)B_{2p}}{(2p)!}\pi^{2p}.$$

Here the numbers B_n are called Bernoulli's numbers, and they are all rational. The first few are

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42},\ldots, \text{ and } \quad B_3=B_5=B_7=\ldots=0,$$

and their generating function is

(4.3) $$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n.$$

These can all be calculated recursively by starting with $B_0=1$, and using

$$\binom{n}{0}B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-1}B_{n-1}=0$$

for $n=2,3,4,\cdots$.

These numbers first appeared in 1713 in Jakob Bernoulli's posthumous book, *The Art of Conjecturing* [3, vol. 3, pp. 164–167]. They arise in a formula which Bernoulli conjectures for the sum $s(p,n)=\sum_{k=1}^{n}k^p$, where p is a natural number.

The Bernoulli numbers were used by Euler in some of his publications, but he does not use them explicitly here. Rather Euler uses consecutive letters of the alphabet in the form $\zeta(2)=A\pi^2$, $\zeta(4)=B\pi^4$, $\zeta(6)=C\pi^6$, … . In these notes we will use the notation $A(n)$ when we refer to Euler's notation so that $A(1)=A$, $A(2)=B$, $A(3)=C$, … . Thus from (4.1) we have Euler's notation in terms of Bernoulli numbers

$$\text{(4.4)}\qquad A(n)=\frac{(-1)^{n+1}2^{2n-1}B_{2n}}{(2n)!}.$$

Euler also summed other alternating series related to $\zeta(z)$. These include

$$L(1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^1}=\frac{\pi}{4},$$

$$L(3)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{32},$$

$$L(5)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}=\frac{\pi^5}{1536}, \text{ and in general}$$

$$L(2p+1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2p+1}}=\frac{(-1)^pE_{2p}}{2^{2p+2}(2p)!}\pi^{2p+1}.$$

(See Knopp [5, page 240].) Here the E_n are called Euler's numbers. They are all integers and the first few are $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385,\ldots$, and $E_1=E_3=E_5=\ldots=0$. The E_{2n} can all be calculated recursively by starting with $E_0=1$, and then using

$$E_{2n}+\binom{2n}{2}E_{2n-2}+\binom{2n}{4}E_{2n-4}+\cdots+E_0=0, \text{ for } n=1,2,3\cdots.$$

<u>Section 6.</u> Euler uses a version of the "Euler Maclaurin summation formula".
In modern times we frequently write this formula as

(6.1)
$$\sum_{k=0}^{n} f(x_0+\alpha k)=\frac{1}{\alpha}\int_{x_0}^{x_0+\alpha n} f(x)dx+\frac{1}{2}\left[f(x_0)+f(x_0+\alpha n)\right]+ \sum_{k=1}^{\infty}\frac{B_{2k}\alpha^{2k-1}}{(2k)!}\left[f^{2k-1}(x_0+\alpha n)-f^{2k-1}(x_0)\right].$$

Usually it appears with $\alpha=1$. We can think of this as a generalization of the trapezoidal rule for numerical integration of the function $f(x)$ over the interval from $x=x_0$ to $x=x_0+\alpha n$ where α is the length of the increment step.

Euler uses a variation of (6.1) which we now describe. Rather than integrating from left to right, we can integrate from right to left. We achieve this modification by replacing the increment α by $-\alpha$, and now think of x_0 is the right most point and $x=x_0-\alpha n$ is the left most point. We get

(6.2)
$$\sum_{k=0}^{n} f(x_0-\alpha n+\alpha k)=\frac{1}{-\alpha}\int_{x_0}^{x_0-\alpha n} f(x)dx+\frac{1}{2}\left[f(x_0)+f(x_0-\alpha n)\right]- \sum_{k=1}^{\infty}\frac{B_{2k}\alpha^{2k-1}}{(2k)!}\left[f^{2k-1}(x_0-\alpha n)-f^{2k-1}(x_0)\right].$$

This form of the summation formula can be easily modified if we want to sum over the infinite range of numbers $x,\ x+\alpha,\ x+2\alpha,\ \cdots$. For this purpose write $x=x_0-\alpha n$ and let $x_0\to\infty$ and $n\to\infty$ in such a way that x remains fixed to get

$$\sum_{k=0}^{\infty} f(x+\alpha k)=\frac{1}{-\alpha}\int_{\infty}^{x} f(x)dx+\frac{1}{2}\left[f(x)+f(\infty)\right]- \sum_{k=1}^{\infty}\frac{B_{2k}\alpha^{2k-1}}{(2k)!}\left[f^{2k-1}(x)-f^{2k-1}(\infty)\right].$$

If we call $\beta(\alpha)=\frac{f(\infty)}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}\alpha^{2k-1}}{(2k)!}f^{2k-1}(\infty)$, then the above summation formula becomes

(6.3)
$$\sum_{k=0}^{\infty} f(x+\alpha k)=\frac{1}{-\alpha}\int_{\infty}^{x} f(x)dx+\frac{1}{2}f(x)-\sum_{k=1}^{\infty}\frac{B_{2k}\alpha^{2k-1}}{(2k)!}f^{2k-1}(x)+\beta(\alpha).$$

Section 7. We must sum alternating series so he modifies (6.3). We replace the increment α by 2α to get

(7.1) $$\sum_{k=0}^{\infty} f(x+2\alpha k) = \frac{1}{-2\alpha}\int_{\infty}^{x} f(x)dx + \frac{1}{2}f(x) - \sum_{k=1}^{\infty}\frac{B_{2k}2^{2k-1}\alpha^{2k-1}}{(2k)!}f^{2k-1}(x) + \beta(2\alpha).$$

Next double (7.1) and subtract the previous summation (6.3) from it to get

(7.1) $$\sum_{k=0}^{\infty}(-1)^k f(x+\alpha k) = \frac{1}{2}f(x) - \sum_{k=1}^{\infty}\frac{B_{2k}\left(2^{2k}-1\right)\alpha^{2k-1}}{(2k)!}f^{2k-1}(x) - \beta(\alpha) + 2\beta(2\alpha).$$

Finally let $\alpha = 1$ and get the general alternating infinite series

$$\sum_{k=0}^{\infty}(-1)^k f(x+k) = \frac{1}{2}f(x) - \sum_{k=1}^{\infty}\frac{B_{2k}\left(2^{2k}-1\right)}{(2k)!}f^{2k-1}(x) - \beta(1) + 2\beta(2).$$

If the series on the left converges, then we see that the constant term $-\beta(1) + 2\beta(2) = 0$ by imagining $x \to \infty$. We get

(7.3) $$\sum_{k=0}^{\infty}(-1)^k f(x+k) = \frac{1}{2}f(x) - \sum_{k=1}^{\infty}\frac{B_{2k}\left(2^{2k}-1\right)}{(2k)!}f^{2k-1}(x).$$

Since Euler does not use the Bernoulli numbers, rather he uses consecutive letters of the alphabet which we describe as $A(n) = \frac{(-1)^{n+1}2^{2n-1}B_{2n}}{(2n)!}$, he writes (6.6) in the form

(7.4) $$\sum_{k=0}^{\infty}(-1)^k f(x+k) = \frac{1}{2}f(x) + \sum_{k=1}^{\infty}\frac{(-1)^k A(k)\left(2^{2k}-1\right)}{2^{2k-1}}f^{2k-1}(x).$$

This is the form of the summation formula used by Euler throughout the remainder of the paper.

To obtain series of the first type Euler sets $f(x) = x^m$ and $\alpha = 1$ in (7.4) to get

$$x^m - (x+1)^m + (x+2)^m - (x+3)^m + (x+4)^m - (x+5)^m + \cdots =$$

$$\frac{1}{2}x^m - \frac{m}{2}\left(2^2-1\right)A(1)x^{m-1} + \frac{m(m-1)(m-2)}{2\cdot\ 2\cdot\ 2}\left(2^4-1\right)A(2)x^{m-3}$$

$$+\frac{m(m-1)(m-2)(m-3)(m-4)}{2\cdot\ 2\cdot\ 2\cdot\ 2\cdot\ 2}\left(2^6-1\right)A(3)x^{m-5} - \cdots,$$

where the sum on the right is finite if *m* is a non-negative integer.

Section 8. Euler now lets $x = 0$ in this last result to obtain series of the first type (1). On the right, all the terms disappear if *m* is even, and all but one term disappears if *m* is odd.

$$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + \cdots$$

(8.1) $$= \begin{cases} 0 & \text{if } m \text{ is even}, \\ \dfrac{(-1)^{(m-1)/2} \; m!}{2^m}\left(2^{m+1}-1\right) A((m+1)/2) & \text{if } m \text{ is odd}. \end{cases}$$

In terms of Bernoulli numbers this is

(8.2) $$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + \cdots = \frac{\left(2^{m+1}-1\right) B_{m+1}}{m+1}$$

.

These sums agree with the results of section 2, but only now is the dependence on $A(n)$ revealed. Comparing (8.2) with (3.2) we have

(8.3) $$\frac{1 - {}_m c_1 + {}_m c_2 - {}_m c_3 + \cdots + (-1)^m \, {}_m c_m}{2^{m+1}} = \frac{\left(2^{m+1}-1\right) B_{m+1}}{m+1} \quad .$$

Hardy [4 , p. 24] gives a derivation of (8.2) that does not use the Euler Maclaurin summation formula and proves that the result is the Abel summation of the divergent series. Hardy begins with the geometric series

$$\frac{x}{x+1} = x - x^2 + x^3 - \cdots \text{ , for } |x| < 1 .$$

Now let $x = e^{-y}$, with $y > 0$ to get

$$\frac{1}{e^y + 1} = e^{-y} - e^{-2y} + e^{-3y} - \cdots .$$

Differentiating m times we get

(8.4) $$\left(-\frac{d}{dy}\right)^m \frac{1}{e^y+1} = 1^m e^{-y} - 2^m e^{-2y} + 3^m e^{-3y} - \cdots ,$$

for $m = 0, 1, 2, \cdots$.

Consider the following elementary manipulations to write $\dfrac{1}{e^y+1}$ as a power series in y in terms of Bernoulli numbers:

$$\frac{1}{e^y+1} = \frac{e^y - 1}{e^{2y} - 1}$$

$$= -\frac{2}{e^{2y}-1} + \frac{e^y+1}{\left(e^y-1\right)\left(e^y+1\right)}$$

$$=-\frac{2}{e^{2y}-1}+\frac{1}{e^{y}-1}$$

$$=\frac{1}{2}-\frac{1}{y}\left[\frac{2y}{e^{2y}-1}\right]-1+\frac{1}{y}\left[\frac{y}{e^{y}-1}\right]+\frac{1}{2}$$

$$=\frac{1}{2}-\frac{1}{y}\left[\frac{2y}{e^{2y}-1}-1+\frac{1}{2}2y\right]+\frac{1}{y}\left[\frac{y}{e^{y}-1}-1+\frac{1}{2}y\right].$$

Using the generating function for the Bernoulli numbers (4.3) we now have

$$\frac{1}{e^{y}+1}=\frac{1}{2}-\frac{1}{y}\sum_{m=2}^{\infty}\frac{B_m}{m!}(2y)^m+\frac{1}{y}\sum_{m=2}^{\infty}\frac{B_m}{m!}y^m \quad .$$

Thus we have shown that

$$\frac{1}{e^{y}+1}=\frac{1}{2}-\sum_{m=1}^{\infty}\frac{2^{m+1}-1}{(m+1)!}B_{m+1}\,y^m \quad .$$

Differentiating m times we get

$$\left(-\frac{d}{dy}\right)^m\frac{1}{e^{y}+1}=\left(-\frac{d}{dy}\right)^m\left(\frac{1}{2}-\sum_{k=1}^{\infty}\frac{2^{k+1}-1}{(k+1)!}B_{k+1}\,y^k\right)$$

$$=(-1)^{m+1}\sum_{k=m}^{\infty}\frac{2^{k+1}-1}{(k+1)!}B_{k+1}\frac{k!}{(k-m)!}y^{k-m} \ .$$

Comparing this with (8.4) we get

$$1^m e^{-y}-2^m e^{-2y}+3^m e^{-3y}-\cdots=(-1)^{m+1}\sum_{k=m}^{\infty}\frac{2^{k+1}-1}{(k+1)!}B_{k+1}\frac{k!}{(k-m)!}y^{k-m} \quad .$$

Write $x=e^{-y}$ to get

(8.5) $$1^m x-2^m x^2+3^m x^3-\cdots=(-1)^{m+1}\sum_{k=m}^{\infty}\frac{2^{k+1}-1}{(k+1)!}B_{k+1}\frac{k!}{(k-m)!}(-\log x)^{k-m} \quad .$$

This last result allows us to find the needed Abel summation of the series. Let $x\to 1$ and only the first term in the series on the right remains. We get

$$1^m-2^m+3^m-\cdots=(-1)^{m+1}\frac{2^{m+1}-1}{m+1}B_{m+1} \quad .$$

Now $B_1=-1/2$, and all the remaining Bernoulli numbers with odd subscripts are zero. Therefore for m = 0 we get

$$1-1+1-1+\cdots=1/2,$$

and for $m=1, 2, 3, \cdots$ we can ignore $(-1)^{m+1}$ since it is negative only when $B_{m+1}=0$. We get

$$(8.6)\qquad 1^m-2^m+3^m-\cdots=\frac{2^{m+1}-1}{m+1}B_{m+1} \ .$$

This last relation is the same as (8.3) and we have rigorously demonstrated the Abel summation of the divergent series $1^m-2^m+3^m-\cdots$ when *m* is a positive integer.

Sections 9. In this section Euler prepares for his main conjecture. When *p* is a positive integer, he has obtained the following two results:

$$1-2^{2p-1}+3^{2p-1}-4^{2p-1}+5^{2p-1}-6^{2p-1}+\cdots=(-1)^{p+1}(2p-1)!\frac{\left(2^{2p}-1\right)}{2^{2p-1}}A(p) \quad \text{, and}$$

$$\frac{1}{1^{2p}}-\frac{1}{2^{2p}}+\frac{1}{3^{2p}}-\frac{1}{4^{2p}}+\frac{1}{5^{2p}}-\frac{1}{6^{2p}}+\cdots=\frac{2^{2p-1}-1}{2^{2p-1}}A(p)\pi^{2p}.$$

Dividing the two he eliminates $A(p)$ and gets

$$(9.1)\qquad \frac{1-2^{2p-1}+3^{2p-1}-4^{2p-1}+5^{2p-1}-6^{2p-1}+\cdots}{\frac{1}{1^{2p}}-\frac{1}{2^{2p}}+\frac{1}{3^{2p}}-\frac{1}{4^{2p}}+\frac{1}{5^{2p}}-\frac{1}{6^{2p}}+\cdots}=\frac{(-1)^{p+1}(2p-1)!\left(2^{2p}-1\right)}{\left(2^{2p-1}-1\right)\pi^{2p}} \ .$$

He also has

$$(9.2)\qquad \frac{1-2^{2p}+3^{2p}-4^{2p}+5^{2p}-6^{2p}+\cdots}{\frac{1}{1^{2p+1}}-\frac{1}{2^{2p+1}}+\frac{1}{3^{2p+1}}-\frac{1}{4^{2p+1}}+\frac{1}{5^{2p+1}}-\frac{1}{6^{2p+1}}+\cdots}=0 \ .$$

Euler lists this pair (9.1) and (9.2) for $p = 1$ to 5.

Section 10. Reflecting on the above results, Euler conjectures the general formula

$$(10.1)\qquad \frac{1-2^{n-1}+3^{n-1}-4^{n-1}+5^{n-1}-6^{n-1}+\cdots}{1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\cdots}=\frac{-(n-1)!\left(2^n-1\right)}{\left(2^{n-1}-1\right)\pi^n}\cos\frac{n\pi}{2} \ .$$

which he has shown to be true for $n = 2, 3, 4, \ldots$. This is the main result of the paper. Today we would write this as

$$(10.2)\qquad \frac{\eta(1-n)}{\eta(n)}=\frac{-\Gamma(n)\left(2^n-1\right)}{\left(2^{n-1}-1\right)\pi^n}\cos\frac{n\pi}{2} \ .$$

Section 11. In this section Euler proves that his conjecture (10.1) is valid for $n = 1$.

Section 12. Now Euler verifies his conjecture (10.1) for $n = 0$. Looking at the conjecture in the form (10.2) we see at once a difficulty since $\Gamma(x)$ has a pole at $x = 0$. This he overcomes by multiplying and dividing by x to get $\frac{\Gamma(x)x}{x} = \frac{\Gamma(x+1)}{x}$. This last expression approaches $\frac{1}{x}$ as $x \to 0$ which simplifies the investigation of the limits that Euler encounters.

Section 13. The conjecture (10.1) has been demonstrated for $n = 0, 1, 2, 3, \ldots$. He now proves the conjecture for n a negative integer. Euler introduces his notation for the generalized factorial. He writes $[x]$ for the modern x!, and so for arbitrary x we have in modern terms $[x] = \Gamma(x+1)$. Looking at the conjecture in the form (10.2) and recalling that $\Gamma(x)$ has poles at $x = 0, 1, 2, \ldots$, we see a problem. This is overcome by using the identity $\Gamma(1-x)\Gamma(1+x) = \frac{\pi x}{\sin \pi x}$, which he has found in a previous publication.

Section 14. Euler verifies his conjecture (10.1) for $n = 1/2$ which is

$$\frac{1 - \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} - \frac{1}{4^{1/2}} + \frac{1}{5^{1/2}} - \frac{1}{6^{1/2}} + \cdots}{1 - \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} - \frac{1}{4^{1/2}} + \frac{1}{5^{1/2}} - \frac{1}{6^{1/2}} + \cdots} = \frac{-\Gamma(1/2)\left(2^{1/2} - 1\right)}{\left(2^{-1/2} - 1\right)\pi^{1/2}} \cos\frac{\pi}{4}.$$

This is the only case considered by Euler in which both the series in the numerator and the denominator converge. He uses $\Gamma(1/2) = \sqrt{\pi}$, and remarks that his success in this case makes his conjecture very convincing.

Section 15. Next Euler tests his conjecture for $n = 3/2$ which is

$$\frac{1 - \sqrt{2} + \sqrt{3} - \sqrt{4} + \sqrt{5} - \sqrt{6} + \cdots}{1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} - \frac{1}{6\sqrt{6}} + \cdots} = \frac{3 + \sqrt{2}}{2\pi\sqrt{2}} = 0.4967738.$$

He makes the verification numerically. This means Euler must find a numerical value for the divergent series in the numerator! A task that would frighten the best analysis. He initially calculates the sum of the first nine terms and gets

$$1 - \sqrt{2} + \sqrt{3} - \sqrt{4} + \sqrt{5} - \sqrt{6} + \sqrt{7} - \sqrt{8} + \sqrt{9} = 1.9217396662.$$

From this he must subtract the remainder of the series which is

$\sqrt{10} - \sqrt{11} + \sqrt{12} - \sqrt{13} + \sqrt{14} - \cdots$ For this second calculation he calls on the alternating series version of the Euler-Maclaurin sum formula (7.3) from which he uses the first 5 terms. This is about 1.541610. Thus the numerator series equals 0.380129.

He calculates the sum of the convergent series in the denominator in the same way (9 terms of the series followed by 5 terms of the Euler Maclaurin formula) and gets 0.765158. Dividing these two values he finds agreement with the conjectured value to 4 decimal places. (Euler actually has 6 decimal place accuracy, but he made a few arithmetical errors.) Euler ends by saying "one could not doubt in the least that this matter is true."

Using Mathematica, we tried the same calculations with the first 60,001 terms of the original series followed by 350 terms of the Euler Maclaurin formula for the remainder of the series. When we compared the numerical results with the conjectured value, we had over 2000 accurate decimal places in less than two minutes.

Sections 17 to 19. Euler continues his search to find the sum of the series

$$\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\frac{1}{7^n}-\frac{1}{8^n}+\cdots,$$

when n is an odd integer. Again he fails. He shows that for λ a positive integer

$$1-\frac{1}{2^{2\lambda+1}}+\frac{1}{3^{2\lambda+1}}-\frac{1}{4^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}-\cdots=$$
$$\frac{2\left(2^{2\lambda}-1\right)\pi^{2\lambda}}{1\cdot 2\cdot 3\cdots 2\lambda\left(2^{2\lambda+1}-1\right)\cos\lambda\pi}(1^{2\lambda}\log 1-2^{2\lambda}\log 2+3^{2\lambda}\log 3-4^{2\lambda}\log 4+\cdots),$$

and observes that summing the series

$$1^{2\lambda}\log 1-2^{2\lambda}\log 2+3^{2\lambda}\log 3-4^{2\lambda}\log 4+etc.$$

is probably more difficult than his original problem.

Section 20. Euler states that he has found the similar conjecture

$$\frac{1-3^{n-1}+5^{n-1}-7^{n-1}+\cdots}{1-3^{-n}+5^{-n}-7^{-n}+\cdots}=\frac{1\cdot 2\cdot 3\cdots(n-1)2^n}{\pi^n}\sin\frac{n\pi}{2}$$

using the same methods.

In modern notion we write

$$L(s)=\frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\cdots=\frac{\zeta(s)+\eta(s)}{2},$$

valid for $\operatorname{Re}(s)>0$. We have the modern functional equation

$$L(1-s) = \frac{\Gamma(s)2^s}{\pi^s}\sin\left(\frac{\pi s}{2}\right)L(s)\ .$$

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