Ramanujan's Notebooks

Part 1

S. Ramanujan, 1919
(From G. H. Hardy, ***Ramanujan,*** *Twelve* ***Lectures on***
Subjects ***Suggested by His*** *Life* ***and Work.***
Cambridge University Press, 1940.)

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Ramanujan's Notebooks

Part 1

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TO
my wife Helen
and
our children Kristin, Sonja, and Brooks

On the Discovery of the Photograph of S. Ramanujan, F.R.S.

S. CHANDRASEKHAR, F.R.S.

Hardy was to give a series of twelve lectures on subjects suggested by Ramanujan's life and work at the Harvard Tercentenary Conference of Arts and Sciences in the fall of 1936. In the spring of that year, Hardy told me that the only photograph of Ramanujan that was available at that time was the one of him in cap and gown, "which make him look ridiculous." And he asked me whether 1 would try to secure, on my next visit to India, a better photograph which he might include with the published version of his lectures. It happened that 1 was in India that same year from July to October. 1 knew that Mrs. Ramanujan was living somewhere in South India, and 1 tried to find where she was living, at first without success. On the day prior to my departure for England in October of 1936, 1 traced Mrs. Ramanujan to a house in Triplicane, Madras. 1 went to her house and found her living under extremely modest circumstances. I asked her if she had any photograph of Ramanujan which 1 might give to Hardy. She told me that the only one she had was the one in his passport which he had secured in London early in 1919. 1 asked her for the passport and found that the photograph was sufficiently good (even after seventeen years) that one could make a negative[1] and copies. It is this photograph which appears in Hardy's book, ***Ramanujan, Twelve Lectures on Subjects Suggested by His Life and Work*** (Cambridge University Press, 1940). It is of interest to recall Hardy's reaction to the photograph: "He looks rather ill (and no doubt was very ill): but he looks all over the genius he was."

[1] It is this photograph which has served as the basis for all later photographs, paintings, etchings, and Paul Granlund's bust of Ramanujan; and the enlargements are copies of the frontispiece in Hardy's book.

from the University Library, Dundee

B. M. Wilson devoted much of his short career to Ramanujan's work. Along with P. V. Seshu Aiyar and G. H. Hardy, he is one of the editors of Ramanujan's *Collected Papers.* In 1929, Wilson and G. N. Watson began the task of editing Ramanujan's notebooks. Partially due to Wilson's premature death in 1935 at the age of 38, the project was never completed. Wilson was in his second year as Professor of Mathematics at The University of St. Andrews in Dundee when he entered hospital in March, 1935 for routine surgery. A blood infection took his life two weeks later. A short account of Wilson's life has been written by H. W. Turnbull [1].

Preface

Ramanujan's notebooks were compiled approximately in the years 1903-1914, prior to his departure for England. After Ramanujan's death in 1920, many mathematicians, including G. H. Hardy, strongly urged that Ramanujan's notebooks be edited and published. In fact, original plans called for the publishing of the notebooks along with Ramanujan's *Collected Papers* in 1927, but financial considerations prevented this. In 1929, G. N. Watson and B. M. Wilson began the editing of the notebooks, but the task was never completed. Finally, in 1957 an unedited photostat edition of Ramanujan's notebooks was published.

This volume is the first of three volumes devoted to the editing of Ramanujan's notebooks. Many of the results found herein are very well known, but many are new. Some results are rather easy to prove, but others are established only with great difficulty. A glance at the contents indicates a wide diversity of topics examined by Ramanujan. Our goal has been to prove each of Ramanujan's theorems. However, for results that are known, we generally refer to the literature where proofs may be found.

We hope that this volume and succeeding volumes will further enhance the reputation of Srinivasa Ramanujan, one of the truly great figures in the history of mathematics. In particular, Ramanujan's notebooks contain new, interesting, and profound theorems that deserve the attention of the mathematical public.

Urbana, Illinois
June, 1984

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Introduction

Srinivasa Ramanujan occupies a central but singular position in mathematical history. The pathway to enduring, meaningful, creative mathematical research is by no means the same for any two individuals, but for Ramanujan, his path was strewn with the impediments of poverty, a lack of a university education, the absence of books and journals, working in isolation in his most creative years, and an early death at the age of 32. Few, if any, of his mathematical peers had to encounter so many formidable obstacles.

Ramanujan was born on December 22, 1887, in Erode, a town in southern India. As was the custom at that time, he was born in the home of his materna1 grandparents. He grew up in Kumbakonam where his father was an accountant for a cloth merchant. Both Erode and Kumbakonam are in the state of Tamil Nadu with Kumbakonam a distance of 160 miles south-southwest of Madras and 30 miles from the Bay of Bengal. Erode lies 120 miles west of Kumbakonam. At the time of Ramanujan's birth, Kumbakonam had a population of approximately 53,000.

Not too much is known about Ramanujan's childhood, although some stories demonstrating his precocity survive. At the age of 12, he borrowed Loney's ***Trigonometry*** [1] from an older student and completely mastered its contents. It should be mentioned that this book contains considerably more mathematics than is suggested by its title. Topics such as the exponential function, logarithm of a complex number, hyperbolic functions, infinite products, and infinite series, especially in regard to the expansions of trigonometric functions, are covered in some detail. But it was Carr's A ***Synopsis of Elementary Results in Pure Mathematics,*** now published under a different title [1], that was to have its greatest influence on Ramanujan. He borrowed this book from the local Government College library at the age of 15 and was thoroughly captivated by its contents. Carr was a tutor at

Cambridge, and his *Synopsis* is a compilation of about 6000 theorems which served as the basis of his tutoring. Much on calculus and geometry but nothing on the theory of functions of a complex variable or elliptic functions is to be found in Carr's book. Ramanujan never learned about functions of a complex variable, but his contributions to the theory of elliptic and modular functions are profound. Very little space in Carr's *Synopsis* is devoted to proofs which, when they are given, are usually very brief and sketchy.

In December, 1903, Ramanujan took the matriculation examination of the University of Madras and obtained a "first class" place. However, by this time, he was completely absorbed in mathematics and would not study any other subject. In particuiar, his failure to study English and physiology caused him to fail his examinations at the end of his first year at the Government College in Kumbakonam. Four years later, Ramanujan entered Pachaiyappa's College in Madras, but again he failed the examinations at the end of his first year.

Not much is known about Ramanujan's life in the years 1903-1910, except for his two attempts to obtain a college education and his marriage in 1909 to Srimathi Janaki. During this time, Ramanujan devoted himself almost entirely to mathematics and recorded his results in notebooks. He also was evidently seriously ill at least once.

Because he was now married, Ramanujan found it necessary to secure employment. So in 1910, Ramanujan arranged a meeting with V. R. Aiyar, the founder of the Indian Mathematical Society. At that time, V. R. Aiyar was a deputy collector in the Madras civil service, and Ramanujan asked him for a position in his office. After perusing the theorems in Ramanujan's notebooks, V. R. Aiyar wrote P. V. Seshu Aiyar, Ramanujan's mathematics instructor while a student at the Government College in Kumbakonam. P. V. Seshu Aiyar, in turn, sent Ramanujan to R. Ramachandra Rao, a relatively wealthy mathematician. The subsequent meeting was eloquently described by R. Ramachandra Rao [1] in his moving tribute to Ramanujan. The content of this memorial and P. V. Seshu Aiyar's [1] sympathetic obituary are amalgamated into a single biography inaugurating Ramanujan's *Collected Papers* [15]. It suffices now to say that R. Ramachandra Rao was indelibly impressed with the contents of Ramanujan's notebooks. He unhesitatingly offered Ramanujan a monthly stipend so that he could continue his mathematical research without worrying about food for tomorrow.

Not wishing to be a burden for others and feeling inadequate because he did not possess a job, Ramanujan accepted a clerical position in the Madras Port Trust Office on February 9, 1912. This was a fortunate event in Ramanujan's career. The chairman of the Madras Port Trust Office was a prominent English engineer Sir Francis Spring, and the manager was a mathematician S. N. Aiyar. The two took a very kindly interest in Ramanujan's welfare and encouraged him to communicate his mathematical discoveries to English mathematicians.

C. P. Snow has revealed, in his engaging collection of biographies [1] and in his foreword to Hardy's book [17], that Ramanujan wrote two English mathematicians before he wrote G. H. Hardy. Snow does not reveal their identities, but A. Nandy [1, p. 147] claims that they are Baker and Popson. Nandy evidently obtained this information in a conversation with J. E. Littlewood. The first named mathematician is H. F. Baker, who was G. H. Hardy's predecessor as Cayley Lecturer at Cambridge and a distinguished analyst and geometer. As Rankin [2] has indicated, the second named by Nandy is undoubtedly E. W. Hobson, a famous analyst and Sadlerian Professor of Mathematics at Cambridge. According to Nandy, Ramanujan's letters were returned to him without comment. The many of us who have received letters from "crackpot" amateur mathematicians claiming to have proved Fermat's last theorem or other famous conjectures can certainly empathize with Baker and Hobson in their grievous errors. Ramanujan also wrote M. J. M. Hill through C. L. T. Griffith, an engineering professor at the Madras Engineering College who took a great interest in Ramanujan's welfare. Rankin [1] has pointed out that Hill was undoubtedly Griffith's mathematics instructor at University College, London, and this was obviously why Ramanujan chose to write Hill. Hill was more sympathetic to Ramanujan's work, but other pressing matters prevented him from giving it a more scrutinized examination. Fortunately, Hill's reply has been preserved; the full text may be found in a compilation edited by Srinivasan [1].

On January 16, 1913, Ramanujan wrote the famed English mathematician G. H. Hardy and "found a friend in you who views my labours sympathetically" [15, p. xxvii]. Upon initially receiving this letter, Hardy dismissed it. But that evening, he and Littlewood retired to the chess room over the commons room at Trinity College. Before they entered the room, Hardy exclaimed that this Hindu correspondent was either a crank or a genius. After $2\frac{1}{2}$ hours, they emerged from the chess room with the verdict-"genius." Some of the results contained in the letter were false, others were well known, but many were undoubtedly new and true. Hardy [20, p. 9] later concluded, about a few continued fraction formulae in Ramanujan's first letter, "if they were not true, no one would have had the imagination to invent them. Finally (you must remember that 1 knew nothing whatever about Ramanujan, and had to think of every possibility), the writer must be completely honest, because great mathematicians are commoner than thieves or humbugs of such incredible skill." Hardy replied without delay and urged Ramanujan to corne to Cambridge in order that his superb mathematical talents might corne to their fullest fruition. Because of strong Brahmin caste convictions and the refusa1 of his mother to grant permission, Ramanujan at first declined Hardy's invitation.

But there was perhaps still another reason why Ramanujan did not wish to sail for England. A letter from an English meteorologist, Sir Gilbert Walker, to the University of Madras helped procure Ramanujan's first officia1 recognition; he obtained from the University of Madras a scholarship of 75 rupees per month beginning on May 1, 1913. Thus, finally, Ramanujan possessed a bona fide academic position that enabled him to devote all of his energy to the pursuit of the prolific mathematical ideas flowing from his creative genius.

At the beginning of 1914, the Cambridge mathematician E. H. Neville sailed to India to lecture in the winter term at the University of Madras. One of Neville's tasks was to convince Ramanujan that he should corne to Cambridge. Probably more important than the persuasions of Neville were the efforts of Sir Francis Spring, Sir Gilbert Walker, and Richard Littlehailes, Professor of Mathematics at Madras. Moreover, Ramanujan's mother consented to her son's wishes to journey to England. Thus, on March 17, 1914, Ramanujan boarded a ship in Madras and sailed for England.

The next three years were happy and productive ones for Ramanujan despite his difficulties in adjusting to the English climate and in obtaining suitable vegetarian food. Hardy and Ramanujan profited immensely from each other's ideas, and it was probably only with a little exaggeration that Hardy [20, p. 11] proclaimed "he was showing me half a dozen new ones (theorems) almost every day." But after three years in England, Ramanujan contracted an illness that was to eventually take his life three years later. It was thought by some that Ramanujan was infected with tuberculosis, but as Rankin [1], [2] has pointed out, this diagnosis appears doubtful. Despite a loss of weight and energy, Ramanujan continued his mathematical activity as he attempted to regain his health in at least five sanatoria and nursing homes. The war prevented Ramanujan from returning to India. But finally it was deemed safe to travel, and on February 27, 1919, Ramanujan departed for home. Back in India, Ramanujan devoted his attention to q-series and produced what has been called his "lost notebook." (See Andrews' paper [2] for a description of this manuscript.) However, the more favorable climate and diet did not abate Ramanujan's illness. On April 26, 1920, he passed away after spending his last month in considerable pain. It might be conjectured that Ramanujan regretted his journey to England where he contracted a terminal illness. However, he regarded his stay in England as the greatest experience of his life, and, in no way, did he blame his experience in England for the deterioration of his health. (For example, see Neville's article [1, p. 295].)

Our account of Ramanujan's life has been brief. Other descriptions may be found in the obituary notices of P. V. Seshu Aiyar [1], R. Ramachandra Rao [1], Hardy [9], [10], [11], [21, pp. 702–720], and P.V. Seshu Aiyar, R. Ramachandra Rao, and Hardy in Ramanujan's *Collected Papers* [15]; the lecture of Hardy in his book ***Ramanujan*** [20, Chapter 1]; the review by Morde11 [1]; an address by Neville [1]; the biographies by Ranganathan [1] and Ram [1]; and the reminiscences in a commemorative volume edited by Bharathi [1].

When Ramanujan died, he left behind three notebooks, the aforementioned "lost notebook" (in fact, a sheaf of approximately 100 loose pages), and other manuscripts. (See papers of Rankin [1] and K. G. Ramanathan [1] for descriptions of some of these manuscripts.) The first notebook was left with Hardy when Ramanujan returned to India in 1919. The second and third notebooks were donated to the library at the University of Madras upon his death. Hardy subsequently gave the first notebook to S. R. Ranganathan, the librarian of the University of Madras who was on leave at Cambridge University for one year. Shortly thereafter, three handwritten copies of all three notebooks were made by T. A. Satagopan at the University of Madras. One copy of each was sent back to Hardy.

Hardy strongly urged that Ramanujan's notebooks be published and edited. In 1923, Hardy wrote a paper [12], [18, pp. 505-516] in which he gave an overview of one chapter in the first notebook. This chapter pertains almost entirely to hypergeometric series, and Hardy pointed out that Ramanujan discovered most of the important classical results in the theory as well as many new theorems. In the introduction to his paper, Hardy remarks that "a systematic verification of the results (in the notebooks) would be a very heavy undertaking." In fact, in unpublished notes left by B. M. Wilson, he reports a conversation with Hardy in which Hardy told him that the writing of this paper [12] was a very difficult task to which he devoted three to four full months of hard work. Original plans called for the notebooks to be published together with Ramanujan's collected published works. However, a lack of funds prevented the notebooks from being published with the *Collected Papers* in 1927.

G. N. Watson and B. M. Wilson agreed in 1929 to edit Ramanujan's notebooks. When they undertook the task, they estimated that it would take them five years to complete the editing. The second notebook is a revised, enlarged edition of the first notebook, and the third notebook has but 33 pages. Thus, they focused their attention on the second notebook. Chapters 2-13 were to be edited by Wilson, and Watson was to examine Chapters 14-21. Unfortunately, Wilson passed away prematurely in 1935 at the age of 38. In the six years that Wilson devoted to the editing, he proved a majority of the formulas in Chapters 2-5, the formulas in the first third of Chapter 8, and many of the results in the first half of Chapter 12. The remaining chapters were essentially left untouched. Watson's interest in the project evidently waned in the late 1930's. Although he examined little in Chapters 14 and 15, he did establish a majority of the results in Chapters 16-21. Moreover, Watson wrote several papers which were motivated by findings in the notebooks.

For several years no progress was made in either the publishing or editing of the notebooks. In 1949, three photostat copies of the notebooks were made at the University of Madras. At a meeting of the Indian Mathematical Society in Delhi in 1954, the publishing of the notebooks was suggested. Finally, in 1957, the Tata Institute of Fundamental Research in Bombay published a photostat edition [16] of the notebooks in two volumes. The first volume reproduces Ramanujan's first notebook, while the second contains the second and third notebooks. However, there is no commentary whatsoever on the contents. The reproduction is very clearly and faithfully executed. If one side of a page is left blank in the notebooks, it is left blank in the facsimile edition. Ramanujan's scratch work is also faithfully reproduced. Thus, on one page we find only the fragment, "If r is positive." The printing was done on heavy, oversized pages with generous margins. Since some pages of the original notebooks are frayed or faded, the photographic reproduction is especially admirable.

Except for Chapter 1, which probably dates back to his school days, Ramanujan began to record his results in notebooks in about 1903. He probably continued this practice until 1914 when he left for England. From biographical accounts, it appears that other notebooks of Ramanujan once existed. It seems likely that these notebooks were preliminary versions of the three notebooks which survive.

The first of Ramanujan's notebooks was written in what Hardy called "a peculiar green ink." The book has 16 chapters containing 134 pages. Following these 16 chapters are approximately 80 pages of heterogeneous unorganized material. At first, Ramanujan wrote on only one side of the page. However, he then began to use the reverse sides for "scratch work" and for recording additional discoveries, starting at the back of the notebook and proceeding forward. Most of the material on the reverse sides has been added to the second notebook in a more organized fashion. The chapters are somewhat organized into topics, but often there is no apparent connection between adjacent sections of material in the same chapter.

The second notebook is, as mentioned earlier, a revised, enlarged edition of the first notebook. Twenty-one chapters comprising 252 pages are found in the second notebook. This material is followed by about 100 pages of disorganized results. In contrast to the first notebook, Ramanujan writes on both sides of each page in the second notebook.

The third notebook contains 33 pages of mostly unorganized material.

We shall now offer some general remarks about the contents of the notebooks. Because the second notebook supersedes the first, unless otherwise stated, all comments shall pertain to the second notebook. The papers of Watson [2] and Berndt [3] also give surveys of the contents.

If one picks up a copy of the notebooks and casually thumbs through the pages, it becomes immediately clear that infinite series abound throughout the notebooks. If Ramanujan had any peers in the forma1 manipulation of infinite series, they were only Euler and Jacobi. Many of the series do not converge, but usually such series are asymptotic series. On only very rare occasions does Ramanujan state conditions for convergence or even indicate that a series converges or diverges. In some instances, Ramanujan indicates that a series (usually asymptotic) diverges by appending the words "nearly" or "very nearly." It is doubtful that Ramanujan possessed a sound grasp of what an asymptotic series is. Perhaps he had never heard of the term "asymptotic." In fact, it was not too many years earlier that the foundations of asymptotic series were laid by Poincaré and Stieltjes. But despite this possible shortcoming, some of Ramanujan's deepest and most interesting results are asymptotic expansions. Although Ramanujan rarely indicated that a series converged or diverged, it is undoubtedly true that Ramanujan generally knew when a series converged and when it did not. In Chapter 6 Ramanujan developed a theory of divergent series based upon the Euler-Maclaurin summation formula. It should be pointed out that Ramanujan appeared to have little interest in other methods of summability, with a couple of examples in Chapter 6 being the only evidence of such interest.

Besides basing his theory of divergent series on the Euler-Maclaurin formula, Ramanujan employed the Euler-Maclaurin formula in a variety of ways. See Chapters 7 and 8, in particular. The Euler-Maclaurin formula was truly one of Ramanujan's favorite tools. Not surprisingly then, Bernoulli numbers appear in several of Ramanujan's formulas. His love and affinity for Bernoulli numbers is corroborated by the fact that he chose this subject for his first published paper [4].

Although series appear with much greater frequency, integrals and continued fractions are plentiful in the notebooks. There are only a few continued fractions in the first nine chapters, but later chapters contain numerous continued fractions. Although Ramanujan is known primarily as a number theorist, the notebooks contain very little number theory. Ramanujan's contributions to number theory in the notebooks are found chiefly in Chapter 5, in the heterogeneous material at the end of the second notebook, and in the third notebook.

The notebooks were originally intended primarily for Ramanujan's own persona1 use and not for publication. Inevitably then, they contain flaws and omissions. Thus, notation is sometimes not explained and must be deduced from the context, if possible. Theorems and formulas rarely have hypotheses attached to them, and only by constructing a proof are these hypotheses discernable in many cases. Some of Ramanujan's incorrect "theorems" in number theory found in his letters to Hardy have been well publicized. Thus, perhaps some think that Ramanujan was prone to making errors. However, such thinking is erroneous. The notebooks contain scattered minor errors and misprints, but there are very few serious errors. Especially if one takes into account the roughly hewn nature of the material and his frequently forma1 arguments, Ramanujan's accuracy is amazing.

On the surface, several theorems in the notebooks appear to be incorrect. However, if *proper* interpretations are given to them, the proposed theorems generally are correct. Especially in Chapters 6 and 8, formulas need to be properly reinterpreted. We cite one example. Ramanujan offers several theorems about $\sum 1/x$, where x is any positive real number. First, we must be aware that, in Ramanujan's notation, $\sum 1/x = \sum_{n \le x} 1/n$. But further reinterpretation is still needed, because Ramanujan frequently intends $\sum 1/x$ to mean $\psi(x + 1) + y$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and y denotes Euler's constant. Recall that if x is a positive *integer*, then $\psi(x + 1) + y = \sum_{n=1}^{x} 1/n$. But in other instances, $\sum 1/x$ may denote Log x + y. Recall that as x tends to ∞, both $\psi(x + 1) + Y$ and $\sum_{n \le x} 1/n$ are asymptotic to Log x + y.

The notebooks contain very few proofs, and those proofs that are given are only very briefly sketched. In contrast to a previous opinion expressed by the author [3], there appear to be more proofs in the first notebook than in the second. They also are more frequently found in the earlier portions of the notebooks; the later chapters contain virtually no indications of proofs. That the notebooks contain few proofs should not be too surprising. First, as mentioned above, the notebooks chiefly served Ramanujan as a compilation of his results; he undoubtedly felt that he could reproduce any of his proofs if necessary. Secondly, paper was scarce and expensive for a poor, uneducated Hindu who had no means of support for many of his productive years. As was the case for most Indian students at that time, Ramanujan worked out most of his mathematics on a slate. One advantage of being employed at the Madras Port Trust Office was that he could use excess wrapping paper for his mathematical research. Thirdly, since Carr's Synopsis was Ramanujan's primary source of inspiration, it was natural that this compendium should serve as a mode1 for compiling his own results.

The nature of Ramanujan's proofs has been widely discussed and debated. Many of his biographers have written that Ramanujan's formulas were frequently inspired by Goddess Namagiri in dreams. Of course, such a view can neither be proved nor disproved. But without discrediting any religious thinking, we adhere to Hardy's opinion that Ramanujan basically thought like most mathematicians. In other words, Ramanujan *proved* theorems like any other serious mathematician. However, his proofs were likely to have severe gaps caused by his deficiencies. Because of the lack of sound, rigorous training, Ramanujan's proofs were frequently formal. Often limits were taken, series were manipulated, or limiting processes were inverted without justification. But, in reality, this might have been one of Ramanujan's strengths rather than a weakness. With a more conventional education, Ramanujan might not have depended upon the original, forma1 methods of which he was proud and rather protective. In particular, Ramanujan's amazingly fertile mind was functioning most creatively in the forma1 manipulation of series. If he had thought like a well-trained mathematician, he would not have recorded many of the formulas which he thought he had proved but which, in fact, he had not proved. Mathematics would be poorer today if history had followed such a course. We are not saying that Ramanujan could not have given rigorous proofs had he had better training. But certainly Ramanujan's prodigious output of theorems would have dwindled had he, with sounder mathematical training, felt the need to provide rigorous proofs by contemporary standards. As an example, we cite Entry 10 of Chapter 3 for which Ramanujan laconically indicated a proof. His "proof," however, is not even valid for any of the examples which he gives to illustrate his theorem. Entry 10 is an extremely beautiful, useful, and deep asymptotic formula for a general class of power series. It would have been a sad loss for mathematics if someone had told Ramanujan to not record Entry 10 because his proof was invalid. Also in this connection, we briefly mention some results in Chapter 8 on analogues of the gamma function. It seems clear that Ramanujan found many of these theorems by working with divergent series. However, Ramanujan's theorems can be proved rigorously by manipulating the series where they converge and then using analytic continuation. Thus, just one concept outside of Ramanujan's repertoire is needed to provide rigorous proofs for these beautiful theorems analogizing properties of the gamma function.

To be sure, there are undoubtedly some instances when Ramanujan did not have a proof of any type. For example, it is well known that Ramanujan discovered the now famous Rogers-Ramanujan identities in India but could not supply a proof until several years later after he found them in a paper of L. J. Rogers, As Littlewood [1], [2, p. 1604] wrote, "If a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further."

In the sequel, we shall indicate Ramanujan's proofs when we have been able to ascertain them from sketches provided by him or from the context in which the theorems appear. We emphasize, however, that for most of his work, we have no idea how Ramanujan made his discoveries. In an interview conducted by P. Nandy [1] in 1982 with Ramanujan's widow S. Janaki, she remarked that her husband was always fearful that English mathematicians would steal his mathematical secrets while he was in England. It seems that not only did English mathematicians not steal his secrets, but generations of mathematicians since then have not discovered his secrets either.

Hardy [20, p. 10] estimated that two-thirds of Ramanujan's work in India consisted of rediscoveries. For the notebooks, this estimate appears to be too high. However, it would be difficult to precisely appraise the percentage of new results in the notebooks. It should also be remarked that some original results in the notebooks have since been rediscovered by others, usually without knowledge that their theorems are found in the notebooks.

Chapter 1 has but 8 pages, while Chapters 2-9 contain either 12 or 14 pages per chapter. The number of theorems, corollaries, and examples in each chapter is listed in the following table.

Chapter	Number of Results
1	43
2	68
3	86
4	50
5	94
6	61
7	110
8	108
9	139
Total	759

In this book, we shall either prove each of these 759 results, or we shall provide references to the literature where proofs may be found. In a few instances, we were unable to interpret the intent of the entries.

In the sequel, we have adhered to Ramanujan's usage of such terms as "corollary" and "example." However, often these designations are incorrect. For example, Ramanujan's "corollary" may be a generalization of the preceding result. An "example" may be a theorem. So that the reader with a copy of the photostat edition of the notebooks can more easily follow our analysis, we have preserved Ramanujan's notation as much as possible. However, in some instances, we have felt that a change in notation is advisable.

Not surprisingly, several of the theorems that Ramanujan communicated in his two letters of January 16, 1913, and February 27, 1913, to Hardy are found in his notebooks. Altogether about 120 results were mailed to Hardy. Unfortunately, one page of the first letter was lost, but all of the remaining theorems have been printed with Ramanujan's collected papers [15]. We have recorded below those results from the letters that are also found in Chapters 1-9 of the second notebook or the Quarterly Reports. Considerably more theorems in Ramanujan's letters were extracted from later chapters in the notebooks.

Location in Collected Papers	Location in Notebooks or Reports
p. xxiv, (2), parts (b), (c)	Chapter 5, Section 30, Corollary 2
p. xxv, IV, (4)	First report, Example (d)
p. xxvi, VI, (1)	Chapter 7, Section 18, Corollary
p. 350, VII, (1)	Chapter 9, Section 27
p. 351, lines 1, 3	Chapter 6, Section 1, Example 2

Many of Ramanujan's papers have their geneses in the notebooks. In all cases, only a portion of the results from each paper are actually found in the notebooks. Also some of the problems that Ramanujan submitted to the *Journal of the Indian Mathematical Society* are ensconced in the notebooks. We list below those papers and problems with connections to Chapters 1-9 or the Quarterly Reports. Complete bibliographic details are found in the list of references.

A condensed summary of Chapters 1-9 will now be provided. More complete descriptions are given at the beginning of each chapter. Because each chapter contains several diverse topics, the chapter titles are only partially indicative of the chapters' contents.

Magic squares can be traced back to the twelfth or thirteenth Century in India and have long been popular amongst Indian school boys. In contrast to the remainder of the notebooks, the opening chapter on magic squares evidently arises from Ramanujan's early school days. Chapter 1 is quite elementary and contains no new insights on magic squares.

Paper	Location in Notebooks
Some properties of Bernoulli numbers	Chapter 5
On question 330 of Prof. Sanjana	Chapter 9, Entries 4(i), (ii)
Irregular numbers	Chapter 5
On the integral $\int_0^x \frac{\tan^{-1} t}{t}\,dt$	Chapter 2, Section 11 Chapter 9, Section 17
On the sum of the square roots of the first n natural numbers	Chapter 7, Section 4, Corollary 4
On the product $\prod_{n=0}^{\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$	Chapter 2, Section 11
Some definite integrals	Chapter 4, Entries 11, 12 Quarterly Reports
Some formulae in the analytic theory of numbers	Chapter 7, Entry 13
Question 260	Chapter 2, Section 4, Corollary
Question 261	Chapter 2, Section 11, Examples 3, 4
Question 327	Chapter 8, Entry 16
Question 386	First Quarterly Report, Example (d)
Question 606	Chapter 9, Section 6, Example (vi)
Question 642	Chapter 9, Section 8

Chapter 2 already evinces Ramanujan's cleverness. Ramanujan examines several finite and infinite series involving arctan x. Especially noteworthy are the curious and fascinating Examples 9 and 10 in Section 5 which follow from ingenious applications of the addition formula for arctan x. The sum

$$\varphi(a) = 1 + 2\sum_{k=1}^{\infty} \frac{1}{(ak)^3 - ak}$$

is examined in detail in Chapter 2 and is revisited in Chapter 8.

Much of Chapter 3 falls in the area of combinatorial analysis, although no combinatorial problems are mentioned. The theories of Bell numbers and single-variable Bell polynomials are developed. It might be mentioned that Bell and Touchard established these theories in print over 20 years after Ramanujan had done this work. Secondly, Ramanujan derives many series expansions that ordinarily would be developed via the Lagrange inversion formula. The method that Ramanujan employed is different and is described in his Quarterly Reports.

Like Chapter 3, Chapter 4 contains essentially two primary topics. First, Ramanujan examines iterates of the exponential function. This material seems to be entirely new and deserves additional study. Secondly, Ramanujan describes an original, formal process of which he was very fond. One of the many applications made by Ramanujan is the main focus of the Quarterly Reports.

Chapter 5 lies in the domain of number theory. Bernoulli numbers, Euler numbers, Eulerian polynomials and numbers, and the Riemann zeta-function $\zeta(s)$ are the chief topics covered. One of the more intriguing results is Entry 29, which, in fact, is false!

Ramanujan's theory of divergent series is set forth in Chapter 6. He associates to each series a "constant." For example, Euler's constant y is the "constant" for the harmonic series. Ramanujan's theory is somewhat flawed but has been put on a firm foundation by Hardy [15].

Chapter 7 continues the subject matter of both Chapters 5 and 6. The functional equation of $\zeta(s)$ is found in disguised form in Entry 4. It is presented in terms of Ramanujan's extended Bernoulli numbers, and his "proof" is based upon his idea of the "constant" of a series. Chapter 7 also contains much numerical calculation.

Analogues of the logarithm of the gamma function form the topic of most of Chapter 8. Ramanujan establishes several beautiful analogues of Stirling's formula, Gauss's multiplication theorem, and Kummer's formula, in particular. Essentially all of this material is original with Ramanujan.

Another analogue of the gamma function is studied in Chapter 9. However, most of the chapter is devoted to the transformation of power series which are akin to the dilogarithm. Although all of Ramanujan's discoveries about the dilogarithm are classical, his many elegant theorems on related functions are generally new. This chapter contains many beautiful series evaluations, some new and some classical.

In 1913, Ramanujan received a scholarship of 75 rupees per month from the University of Madras. A stipulation in the scholarship required Ramanujan to write quarterly reports detailing his research. Three such reports were written before he departed for England, and they have never been published. This volume concludes with an analysis of the content of the quarterly reports. The first two reports and a portion of the third are concerned with a type of interpolation formula in the theory of integral transforms, which is original and is discussed in Chapter 4. However, in the reports, Ramanujan discusses his theorem in much greater detail, provides a "proof," and gives numerous examples in illustration. His most noteworthy new finding is a broad generalization of Frullani's integral theorem that has not been heretofore observed. Using a sort of converse theorem to his interpolation formula, Ramanujan derives many unusual series expansions.

We collect now some notation and theorems that will be used several times in the sequel. We shall not employ the conventions used by Ramanujan for the Bernoulli numbers B_n, $0 \le n < \infty$, but instead we employ the contemporary definition found, for example, in the compendium of Abramowitz and Stegun [1, p. 804], i.e.,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \qquad |x| < 2n. \tag{I1}$$

We adhere to the current convention for the Euler numbers E_n, $0 \le n < \infty$;

thus, $E_{2n+1} = 0$, $n \geq 0$, while E_{2n}, n $\geq$ 0, is defined by

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}, \qquad |x| < \frac{\pi}{2}, \tag{I2}$$

which again differs from the convention used by Ramanujan.

Many applications of the Euler-Maclaurin summation formula will be made. Versions of the Euler-Maclaurin formula may be found in the treatises of Bromwich [1, p. 328], Knopp [1, p. 524], and Hardy [15, Chapter 133, for example. If f has $2n + 1$ continuous derivatives on [a, β], where a and β are integers, then

$$\sum_{k=\alpha}^{\beta} f(k) = \int_{\alpha}^{\beta} f(t)\,dt + \tfrac{1}{2}\{f(\alpha) + f(\beta)\} + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}\{f^{(2k-1)}(\beta) - f^{(2k-1)}(\alpha)\} + R_n, \tag{I3}$$

where, for $n \geq 0$,

$$R_n = \frac{1}{(2n+1)!} \int_{\alpha}^{\beta} B_{2n+1}(t - [t]) f^{(2n+1)}(t)\,dt, \tag{I4}$$

where B,(x), $0 \leq n < \infty$, denotes the nth Bernoulli polynomial. For brevity, we sometimes put P,(x) = $B_n(x - [x])/n!$. In the sequel, we shall frequently let β = x, where x is to be considered large. Letting n tend to ∞ in (13) then normally produces an asymptotic series as x tends to ∞. In these instances, we shall write (13) in the form

$$\sum_{k=\alpha}^{x} f(k) \sim \int_{\alpha}^{x} f(t)\,dt + c + \tfrac{1}{2} f(x) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x), \tag{I5}$$

as x tends to ∞, where c is a certain constant.

As usual, Γ denotes the gamma function. Recall Stirling's formula,

$$\Gamma(x+1) \sim \sqrt{2\pi}\, x^{x+1/2} e^{-x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right), \tag{I6}$$

as x tends to ∞. (See, for example, Whittaker and Watson's text [1, p. 253].) At times, we shall employ the shifted factorial

$$(a)_k = a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \tag{I7}$$

where k is a nonnegative integer.

In the sequel, equation numbers refer to equations in that chapter in which reference is made, except for two types of exceptions. The equalities in the Introduction are numbered (Il), (I2), etc. Secondly, when an equation from another chapter is used, that chapter will be specified.

In referring to the notebooks, the pagination of the Tata Institute will be employed. Unless otherwise stated, page numbers refer to volume 2.

Because of the unique circumstances shaping Ramanujan's career, inevitable queries arise about his greatness. Here are three brief assessments of Ramanujan and his work.

Paul Erdös has passed on to us Hardy's persona1 ratings of mathematicians. Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100. Hardy gave himself a score of 25, Littlewood 30, Hilbert 80, and Ramanujan 100.

Neville [1] began a broadcast in Hindustani in 1941 with the declaration, "Srinivasa Ramanujan was a mathematician SO great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years."

In notes left by B. M. Wilson, he tells us how George Polya was captivated by Ramanujan's formulas. One day in 1925 while Polya was visiting Oxford, he borrowed from Hardy his copy of Ramanujan's notebooks. A couple of days later, Polya returned them in almost a state of panic explaining that however long he kept them, he would have to keep attempting to verify the formulae therein and never again would have time to establish another original result of his own.

TO be sure, India has produced other great mathematicians, and Hardy's views may be moderately biased. But even though the pronouncements of Neville and Hardy are overstated, the excess is insignificant, for Ramanujan reached a pinnacle scaled by few. It is hoped that readers of our analyses of Ramanujan's formulas will be captivated by them as Polya once was and will join the chorus of admiration along with Hardy, Neville, Polya, and countless others.

The task of editing Ramanujan's second notebook has been greatly facilitated by notes left by B. M. Wilson. Accordingly, he has been listed as a coauthor on earlier published versions of Chapters 2-5 to which he made extensive contributions. Wilson's notes were given to G. N. Watson upon Wilson's death in 1935. After Watson passed away in 1965, his papers, including Wilson's notes, were donated to Trinity College, Cambridge, at the suggestion of R. A. Rankin. We are grateful to the Master and Fellows of Trinity College for a copy of Watson and Wilson's notes on the notebooks and for permission to use these notes in our accounts.

We sincerely appreciate the collaboration of Ronald J. Evans on Chapters 3 and 7 and Padmini T. Joshi on Chapters 2 and 9. The accounts of the aforementioned chapters are superior to what the author would have produced without their contributions. Versions of Chapters 2-9 and the Quarterly Reports have appeared elsewhere. We list below the publications where these papers appeared.

We appreciate very much the help that was freely given by several people as we struggled to interpret and prove Ramanujan's findings. D. Zeilberger provided some very helpful suggestions for Chapters 3 and 4. The identities of others are related in the following chapters. However, we particularly draw attention to Richard A. Askey and Ronald J. Evans. Askey carefully read our

Chapter	Coauthors	Publication
2	P. T. Joshi, B. M. Wilson	Glasgow Math. *J.*, 22 (1981), 199-216.
3	R. J. Evans, B. M. Wilson	*Adv. Math.,* ***49*** (1983), 123-169.
4	B. M. Wilson	*Proc. Royal Soc. Edinburgh,* **89A** (1981), 87–109.
5	B. M. Wilson	***Analytic Number Theory*** (M. I. Knopp, ed.). Lecture Notes in Math., No. 899, Springer-Verlag, Berlin, 198 1, pp. 49978.
6		***Resultate der Math., 6*** (1983), *l-26.*
7	R. J. Evans	*Math. Proc. Nat. Acad. Sci. India,* ***92*** (1983), 67-96.
8		*J.* ***Reine Angew. Math., 338*** (1983), l-55.
9	P. T. Joshi	***Contemporary Mathematics,*** vol. 23, Amer. Math. Soc., Providence, 1983.
Quarterly Reports (Abridged Version)		***Amer. Math.*** *Monthly,* ***90*** (1983), 505–516.
Quarterly Reports		***Bull. London Math. Soc.,*** 16 (1984), 4499489.

manuscripts and offered many suggestions, references, and insights. Evans proved some of Ramanujan's deepest and most difficult theorems.

The manuscript was typed by the three best technical typists in Champaign-Urbana, Melody Armstrong, Hilda Britt, and Dee Wrather. We thank them for the superb quality of their typing.

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CHAPTER 1

Magic Squares

The origin of Chapter 1 probably is found in Ramanujan's early school days and is therefore much earlier than the remainder of the notebooks. Rules for constructing certain rectangular arrays of natural numbers are given. Most of Ramanujan's attention is devoted to constructing magic squares. A magic square is a square array of (usually distinct) natural numbers so that the sum of the numbers in each row, column, or diagonal is the same. In some instances, the requirement on the two diagonal sums is dropped. In the notebooks, Ramanujan uses the word "corner" for "diagonal." We emphasize that the theory of magic squares is barely begun by Ramanujan in Chapter 1. Considerably more extensive developments are contained in the books of W. S. Andrews [1] and Stark [1], for example.

Ramanujan commences Chapter 1 with the following simple principle for constructing magic squares. Consider two sets of natural numbers $S_1 = \{A, B, C, \ldots\}$ and $S_2 = \{P, Q, R, \ldots\}$. each with n elements. Take the n^2 numbers in the direct sum $S_1 + S_2$ and arrange them in an $n \times n$ square so that each letter appears exactly once in each row, column, and diagonal. Clearly, we have then constructed a magic square. Of course, some numbers may appear more than once.

In Corollary 1, Ramanujan states the trivial fact that if $A + P$, $A + Q$, $A + R, \ldots$ are in arithmetic progression, then $B + P$, $B + Q$, $B + R, \ldots$ are also in arithmetic progression.

In Corollary 2, Ramanujan remarks that if $A + P$, $A + Q$, $A + R$, $A + S, \ldots$ are known and also $B + P$ is given, then we can determine $B + Q$, $B + R$, $B + S, \ldots$. This is clear, for $B-A$ is thus known, and we may write $B + Q = (B - A) + (A + Q)$, etc.

Ramanujan informs us that in constructing a magic square, we should not give values to $A, B, C, \ldots$ and $P, Q, R, \ldots$ but instead values should be assigned to $A+P, A+Q, A+R, \ldots$. The reason for this advice is not clear, for in either case $2n$ parameters need to be prescribed.

Example 1. *Given that* $A+P=8$, $B+P=10$, $C+P=11$, $D+P=14$, *and* $C+R=25$, *find* $A+R$, $B+R$, *and* $D+R$.

Solution. Since $R-P=14$, then $A+R=(A+P)+(R-P)=22$. Similarly, $B+R=24$ and $D+R=28$.

Example 2. *Given* $A+P=5$, $A+Q=7$, $A+S=17$, $B+Q=23$, *and* $B+R=26$, *find* $A+R$, $B+P$, *and* $B+S$.

Solution. Note that $B-A=16$. Hence, $A+R=-16+26=10$, $B+P=21$, and $B+S=33$.

Entry 2(i). *Let m_1 and m_2 denote the sums of the middle row and middle column, respectively, of a 3×3 square array of numbers. Let c_1 and c_2 denote the sums of the main diagonal and secondary diagonal, respectively. Lastly, let S denote the sum of all nine elements of the square. Then if x denotes the center element of the square,*

$$x=\tfrac{1}{3}(m_1+m_2+c_1+c_2-S).$$

Proof. Observe that

$$m_1+m_2+c_1+c_2=S+3x,$$

since x is counted four times on the left side. The result now follows.

Entry 2(ii). *Suppose that the sum of each row and column is equal to r. Then, in the notation of Entry* 2(i),

$$x=\tfrac{1}{3}(c_1+c_2-r).$$

Proof. By Entry 2(i).

$$x=\tfrac{1}{3}(r+r+c_1+c_2-3r)=\tfrac{1}{3}(c_1+c_2-r).$$

Note that if the square is magic, then Entry 2(ii) implies that $x=r/3$, and so r is a multiple of 3.

Corollary 1. *In a 3 x 3 magic square, the elements in the middle row, middle column, and each diagonal are in arithmetic progression.*

Proof. In each case, the second element is equal to $r/3$ by the remark above. If a and b are the first and third elements, respectively, in any of the four cases, then

$$a+r/3+b=r.$$

Hence,

$$b - r/3 = r/3 - a,$$

i.e., the three numbers are in arithmetic progression.

Example 1. *Construct magic squares with* (i) $r = 15$, *and* (ii) Y = 27 *and all numbers odd.*

Solutions.

6	1	8
7	5	3
2	9	4

15	1	11
5	9	13
7	17	3

Example *2. Construct magic squares with* (i) $r = 36$ *and all elements even, and* (ii) $r = 63$ *and all elements divisible by 3.*

Solutions.

14	4	18
16	12	8
6	20	10

24	9	30
27	21	15
12	33	18

Ramanujan begins Section 4 with a general construction for a 3 x 3 magic square:

$C+Q$	$A+P$	$B+R$
$A+R$	$B+Q$	$C+P$
$B+P$	$C+R$	$A+Q$

For this square to actually be magic, it is easily seen that A, B, C and P, Q, R must each be arithmetic progressions. Adjacent to the square above, there is an unexplained 4 x 4 square partially filled with the marks A, V, and x .

Example 1(i). *Construct a 3 x 3 square with all row and column sums equal to* **19** *but with only one diagonal sum equal to* **19.**

Solution.

10	2	7
4	6	9
5	11	3

Example 1(ii). *Construct a 3 x 3 square with all row and column sums equal to 31 but with dnly one diagonal sum equal to* **31.** *Ramanujan also requires that all the elements be odd, but the example that he gives does not satisfy this criterion.*

Solution.

14	5	12
7	11	13
10	15	6

Example 2(i). *Construct a 3 x 3 square with all row and column sums equal to 20 and diagonal sums equal to* **16** *and* **19.**

Solution.

10	2	8
4	5	11
6	13	1

Example 2(ii). *Construct a square with diagonal sums* **15** *and* **19,** *column sums* **16, 17,** *and* **12,** *and row sums 6,* **21,** *and* **18.**

Solution.

1	2	3
8	9	4
7	6	5

In Section 5, Ramanujan turns his attention to the construction of certain rectangles which he calls "oblongs." First, he gives the following general construction of a 3 x 4 rectangle with equal row sums and with equal column sums:

A	$C+D$	$A+2D$	$C+3D$
$B+6D$	$B+4D$	$B+2D$	B
C	$A+D$	$C+2D$	$A+3D$

In order for this rectangle to satisfy the designated specifications, we need to require that $A + C = 2B + 3D$. The common row sum will then be equal to $A + C + 2B + 9D$. Adjacent to the rectangle displayed above, there appears an unexplained 3 x 4 rectangle filled with the symbols Λ, v , and x .

Example. *Construct 3 x 4 rectangles where the average of the elements in each row and column is equal to* (i) *8, and* (ii) 15 *with all numbers odd.*

Solutions.

1	13	3	15
11	9	7	5
12	2	14	4

1	25	5	29
21	17	13	9
23	3	27	7

Observe that the requirement of *average* row and column sums in a rectangle is the correct analogue of *equal* row and column sums in a square.

Section 6 is devoted to the construction of 4 x 4 magic squares. Ramanujan begins with the easily ascertained equality,

$$\begin{aligned}\text{sum of middle 4 elements} = \tfrac{1}{2}(&\text{sum of diagonals}\\ &+ \text{sum of middle rows}\\ &+ \text{sum of middle columns} - \text{total sum}),\end{aligned}$$

except that Ramanujan has the wrong sign on the left side.

Entry 6(ii). *Construct a magic square by letting* $S_1 = \{A, B, C, D\}$ *and* $S_2 = \{P, Q, R, S\}$ *and considering* $S_1 + S_2$.

Solutions.

$A+P$	$D+S$	$C+Q$	$B+R$
$C+R$	$B+Q$	$A+S$	$D+P$
$B+S$	$C+P$	$D+R$	$A+Q$
$D+Q$	$A+R$	$B+P$	$C+S$

$A+P$	$D+Q$	$D+R$	$A+S$
$B+S$	$C+R$	$C+Q$	$B+P$
$C+S$	$B+R$	$B+Q$	$C+P$
$D+P$	$A+Q$	$A+R$	$D+S$

There are no restrictions on the elements in the first square, but in the second we need to require that $A + D = B + C$ and $P + S = Q + R$.

In a note, Ramanujan remarks "If $A + D = B + C$ and $P + R = Q + S$ the extreme middle four in the first square also satisfy the given condition." "The extreme middle four" is not defined by Ramanujan, but presumably they are the four middle squares which, in fact, have been blocked out by Ramanujan. But then the hypotheses $A + D = B + C$ and $P + R = Q + S$ are not needed!

Example 1. *Construct 4 x 4 magic squares with common sums of 34, 34, and 35.*

Solution.

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

1	15	11	8
12	7	2	14
6	9	17	3
16	4	5	10

All three examples are instances of the first general construction described above. A table of parameters for these three examples as well as the next two examples is provided below.

Example	A	B	C	D	P	Q	R	S
1a	1	5	9	13	0	2	3	1
1b	1	3	7	5	0	8	1	9
1c	1	5	9	14	0	2	3	1
2a	1	5	25	29	0	2	3	1
2b	9	13	17	21	0	2	3	1

Example 2. *Construct two 4 x 4 magic squares with common sums of 66.*

Solution.

1	30	27	8
28	7	2	29
6	25	32	3
31	4	5	26

9	22	19	16
20	15	10	21
14	17	24	11
23	12	13	18

Example 3. *Construct two 3 x 3 magic squares with common sum 60.*

Solution.

28	1	31
23	20	17
9	39	12

25	3	32
27	20	13
8	37	15

Ramanujan commences Section 7 by correctly asserting that a magic square of m rows can be formèd from magic squares of n rows when $n|m$, except in one case. This case is when $m = 6$ and $n = 3$. In this instance, each of the four magic squares of three rows must have the same common sum r. The center element in each square is then $r/3$, contradicting the requirement that the elements be distinct. However, 6 x 6 magic squares can be constructed from 3 x 3 squares if the requirement that the diagonal sums be the same as the row and column sums is relaxed in the construction of the four 3 x 3 squares.

Ramanujan now gives two examples of 8 x 8 magic squares. The first is constructed from four 4 x 4 magic squares, while the second is not.

1	62	59	8	9	54	51	16
60	7	2	61	52	15	10	53
6	57	64	3	14	49	56	11
63	4	5	58	55	12	13	50
17	46	43	24	25	38	35	32
44	23	18	45	36	31	26	37
22	41	48	19	30	33	40	27
47	20	21	42	39	28	29	34

1	58	59	4	5	62	63	8
16	55	54	13	12	51	50	9
24	47	46	21	20	43	42	17
25	34	35	28	29	38	39	32
33	26	27	36	37	30	31	40
48	23	22	45	44	19	18	41
56	15	14	53	52	11	10	49
57	2	3	60	61	6	7	64

Ramanujan begins Section 8 by once again enunciating the method for constructing magic squares described in Sections 1, 3, and 6. He offers two general constructions of 5 x 5 magic squares; namely

$A+P$	$E+R$	$D+T$	$C+Q$	$B+S$
$C+T$	$B+Q$	$A+S$	$E+P$	$D+R$
$E+S$	$D+P$	$C+R$	$B+T$	$A+Q$
$B+R$	$A+T$	$E+Q$	$D+S$	$C+P$
$D+Q$	$C+S$	$B+P$	$A+R$	$E+T$

$D+Q$	$E+S$	$A+P$	$B+R$	$C+T$
$E+R$	$A+T$	$B+Q$	$C+S$	$D+P$
$A+S$	$B+P$	$C+R$	$D+T$	$E+Q$
$B+T$	$C+Q$	$D+S$	$E+P$	$A+R$
$C+P$	$D+R$	$E+T$	$A+Q$	$B+S$

There are no restrictions on the parameters in the first square, but in the second, the condition $A + B + D + E = 4C$ must be satisfied.

Example 1. *Construct 5 x 5 magic squares with common sums of 65 and 66.*

Solution.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

1	24	20	12	9
15	7	4	22	18
25	16	13	10	2
8	5	23	19	11
17	14	6	3	26

The first magic square arises from the second general construction and, according to W. S. Andrews [1, p. 2], is a very old magic square. The second is a consequence of the first general method. The parameters may be chosen by taking $P=0$, $Q=1$, $R=2$, $S=3$, $T=4$, $A=1$, $B=6$, $C=11$, $D=16$, and E = 21, 22, respectively.

Example 2. *Construct 7 x 7 magie squares with common sums of* 170 *and* 175.

Solution.

1	26	45	15	41	12	30
38	9	34	5	23	42	19
27	46	16	35	13	31	2
10	28	6	24	43	20	39
47	17	36	14	32	3	21
29	7	25	44	14	40	11
18	37	8	33	4	22	48

1	49	41	33	25	17	9
18	10	2	43	42	34	26
35	27	19	11	3	44	36
45	37	29	28	20	12	4
13	5	46	38	30	22	21
23	15	14	6	47	39	31
40	32	24	16	8	7	48

CHAPTER 2

Sums Related to the Harmonic Series or the Inverse Tangent Function

Chapter 2 is fairly elementary, but several of the formulas are very intriguing and evince Ramanujan's ingenuity and cleverness. Ramanujan gives more proofs in this chapter than in most of the later chapters.

Many of the formulas found herein are identities between finite sums. Many of these identities involve arctan x, and because this function arises so frequently in the sequel, we shall put $A(x) = \arctan x$. It will be assumed that $-\pi/2 \leq A(x) \leq \pi/2$. Several of Ramanujan's theorems concerning this function arise from the elementary equalities

$$A(x) + A(y) = A\left(\frac{x+y}{1-xy}\right), \tag{0.1}$$

except when $xy > 1$, and

$$A(x) - A(y) = A\left(\frac{x-y}{1+xy}\right), \tag{0.2}$$

except when $-xy > 1$.

Entries 1, 2, 4, 5, and 6 involve the functions

$$\varphi(a, n) = 1 + 2\sum_{k=1}^{n} \frac{1}{(ak)^3 - ak}$$

and $\varphi(a) = \lim_{n\to\infty} \varphi(a, n)$, where a is an integer exceeding one. Ramanujan continues his study of $\varphi(a)$ in Chapter 8.

Entry 1. *For each positive integer n,*

$$\sum_{k=1}^{n} \frac{1}{n+k} = \frac{n}{2n+1} + \sum_{k=1}^{n} \frac{1}{(2k)^3 - 2k} \tag{1.1}$$

Proof. We give Ramanujan's proof. In the easily verified identity

$$\frac{1}{x^3-x}=\frac{1}{2(x-1)}+\frac{1}{2(x+1)}-\frac{1}{x}, \tag{1.2}$$

let $x = 2k$ and sum on k, $1 \le k \le n$. The right side of (1.1) is then found to be equal to

$$\frac{1}{2}\sum_{k=1}^{n}\frac{1}{2k-1}+\frac{1}{2}\sum_{k=1}^{n}\frac{1}{2k+1}-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}+\frac{n}{2n+1}$$

$$=\sum_{k=1}^{n}\frac{1}{2k-1}-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}$$

$$=\sum_{k=1}^{2n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k}=\sum_{k=1}^{n}\frac{1}{n+k}. \tag{1.3}$$

Corollary. $$\operatorname{Log} 2=\tfrac{1}{2}\varphi(2).$$

Proof. Using the following well-known fact found in Ayoub's text [1, p. 433],

$$\lim_{x\to\infty}\left\{\sum_{k\le x}\frac{1}{k}-\operatorname{Log} x\right\}=\gamma, \tag{1.4}$$

where γ denotes Euler's constant, we find from the last equality in (1.3) that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to\infty}\left\{\left(\sum_{k=1}^{2n}\frac{1}{k}-\operatorname{Log}(2n)\right)-\left(\sum_{k=1}^{n}\frac{1}{k}-\operatorname{Log} n\right)\right\}+\operatorname{Log} 2$$

$$=\operatorname{Log} 2.$$

The result now follows from Entry 1 and the definition of $\varphi(a)$.

There is a different proof of this corollary in Ramanujan's first notebook (vol. 1, p. 7). This proof is also discussed in the author's paper [3, p. 154].

Example. *For each positive integer n,*

$$\sum_{k=1}^{n}\frac{n-k}{n+k}=2n\sum_{k=1}^{n}\frac{1}{(2k-1)2k(2k+1)}-\frac{n}{2n+1}.$$

Proof. The proof below is given by Ramanujan. Multiply both sides of (1.1) by $2n$ to get

$$\sum_{k=1}^{n}\frac{2n}{n+k}=2n\sum_{k=1}^{n}\frac{1}{(2k-1)2k(2k+1)}+\frac{2n^2}{2n+1}. \tag{1.5}$$

Subtract 1 from each term on the left side of (1.5) and n from the right side of (1.5) to achieve the desired equality.

Entry 2. *For each positive integer n,*

$$\sum_{k=1}^{2n+1} \frac{1}{n+k} = \varphi(3, n). \tag{2.1}$$

Proof. Using (1.2), we find that

$$\varphi(3, n) = 1 + 2\sum_{k=1}^{n} \left\{ \frac{1}{2(3k-1)} + \frac{1}{2(3k+1)} - \frac{1}{3k} \right\}$$

$$= \sum_{k=1}^{3n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k},$$

from which the desired result follows.

Corollary. $\text{Log } 3 = \varphi(3).$

Proof. The proof is like that of the corollary to Entry 1. Let n tend to ∞ in (2.1) and use the fact that, by (1.4),

$$\lim_{n\to\infty} \sum_{k=n+1}^{3n+1} \frac{1}{k} = \text{Log } 3. \tag{2.2}$$

Ramanujan's proof of the corollary above is similar to the aforementioned proof that he gave in the first notebook for the corollary of Entry 1. Ramanujan replaced n by $1/dx$ in the left side of (2.1) and regarded this sum as a Riemann sum. Thus,

$$\lim_{dx\to 0} \sum_{k=1}^{2n+1} \frac{dx}{1+k\,dx} = \int_1^3 \frac{dx}{x} = \text{Log } 3,$$

from which the corollary follows.

Entry 3. *For each positive integer n,*

$$\sum_{k=1}^{2n+1} A\left(\frac{1}{n+k}\right) = \frac{\pi}{4} + \sum_{k=1}^{n} A\left(\frac{10k}{(3k^2+2)(9k^2-1)}\right). \tag{3.1}$$

Proof. By (0.1) and (0.2), respectively,

$$A\left(\frac{1}{3k-1}\right) + A\left(\frac{1}{3k+1}\right) = A\left(\frac{6k}{9k^2-2}\right) \tag{3.2}$$

and

$$A\left(\frac{1}{k}\right) - A\left(\frac{1}{3k}\right) = A\left(\frac{2k}{3k^2+1}\right), \tag{3.3}$$

for each positive integer k. By (3.2), (3.3), and (0.2), we find that

$$A\left(\frac{1}{3k-1}\right)+A\left(\frac{1}{3k}\right)+A\left(\frac{1}{3k+1}\right)-A\left(\frac{1}{k}\right)=A\left(\frac{10k}{(3k^2+2)(9k^2-1)}\right). \tag{3.4}$$

If we now sum both sides of (3.4) for $1 \le k \le n$, we readily complete the proof of Entry 3.

Note that, by (3.1) and Taylor's theorem,

$$\sum_{k=1}^{n} A\left(\frac{10k}{(3k^2+2)(9k^2-1)}\right)=\sum_{k=n+1}^{3n+1}\left\{\frac{1}{k}+O\left(\frac{1}{k^3}\right)\right\}-\frac{\pi}{4}$$
$$=\sum_{k=n+1}^{3n+1}\frac{1}{k}+O\left(\frac{1}{n^2}\right)-\frac{\pi}{4}.$$

Letting n tend to ∞ and using (2.2), we deduce that

$$\sum_{k=1}^{\infty} A\left(\frac{10k}{(3k^2+2)(9k^2-1)}\right)=\operatorname{Log} 3-\frac{\pi}{4},$$

which is given by Ramanujan in his first notebook (vol. 1, p. 9).

Entry 4. *For each positive integer n,*

$$\sum_{k=1}^{n}\frac{1}{n+k}+\sum_{k=0}^{n}\frac{1}{2n+2k+1}=\varphi(4,n)=\sum_{k=1}^{4n+1}\frac{(-1)^{k+1}}{k}+\frac{1}{2}\sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k}. \tag{4.1}$$

Proof. The complete proof is given by Ramanujan. By (1.2),

$$\varphi(4,n)=1+\sum_{k=1}^{n}\left\{\frac{1}{4k-1}+\frac{1}{4k+1}-\frac{1}{2k}\right\}$$
$$=\sum_{k=1}^{4n+1}\frac{1}{k}-\frac{1}{2}\sum_{k=1}^{2n}\frac{1}{k}-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}$$
$$=\sum_{k=1}^{3n+1}\frac{1}{n+k}-\sum_{k=1}^{n}\frac{1}{2n+2k}$$
$$=\sum_{k=1}^{n}\frac{1}{n+k}+\sum_{k=0}^{n}\frac{1}{2n+2k+1}, \tag{4.2}$$

which proves the first equality in (4.1).

Next, using the second equality in (4.2), we find that

$$\varphi(4, n) = \sum_{k=1}^{4n+1} \frac{1}{k} - 2\sum_{k=1}^{2n} \frac{1}{2k} + \frac{1}{2}\sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2k}$$

$$= \sum_{k=1}^{4n+1} \frac{(-1)^{k+1}}{k} \cdot + \frac{1}{2}\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k},$$

which establishes the second equality in (4.1).

In the **proof** above, and elsewhere, Ramanujan frequently uses a rather unorthodox notation. Thus, for example,

$$\sum \frac{1}{2n} \text{ means } \sum_{k=1}^{2n} \frac{1}{k} \quad \text{and} \quad \sum \frac{1}{4n+1} \text{ means } \sum_{k=1}^{4n+1} \frac{1}{k}.$$

The corollary below represents the first problem that Ramanujan **sub-**mitted to the *Journal of the* ***Indian Mathematical*** *Society* [1], [15, p. 322]. Ironically, this result was previously posed as a problem by **Lionnet** [1] in 1879. The problem and its solution are also given in Chrystal's textbook [1, p. 249].

Corollary. $\frac{3}{2}$ Log 2 = (p(4).

***Proof*.** Let n tend to cc on the right **side** of (4.1) and use the equality $\sum_{k=1}^{\infty} (-1)^{k+1}/k = \text{Log } 2$.

Entry 5. *For euch positive integer n,*

$$\varphi(6, n) = \frac{2}{3}\sum_{k=1}^{n} \frac{1}{n+k} + \sum_{k=0}^{2n} \frac{1}{2n+2k+1}. \tag{5.1}$$

***Proof*.** By (1.2),

$$\varphi(6, n) = 1 + \sum_{k=1}^{n} \left\{\frac{1}{6k-1} + \frac{1}{6k+1} - \frac{1}{3k}\right\}$$

$$= \sum_{k=1}^{6n+1} \frac{1}{k} - \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \left\{\frac{1}{6k-4} + \frac{1}{6k-3} + \frac{1}{6k-2} + \frac{1}{6k}\right\}$$

$$= \sum_{k=1}^{6n+1} \frac{1}{k} - \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{3n} \frac{1}{k} - \frac{1}{3}\sum_{k=1}^{n} \frac{1}{2k-1}$$

$$= \sum_{k=1}^{6n+1} \frac{1}{k} - \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{3n} \frac{1}{k} - \frac{1}{3}\left\{\sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}\right\}$$

$$= \sum_{k=1}^{6n+1} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{3n} \frac{1}{k} - \frac{1}{3}\sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n} \frac{1}{k}$$

$$= \sum_{k=n+1}^{6n+1} \frac{1}{k} - \frac{1}{2} \sum_{k=n+1}^{3n} \frac{1}{k} - \frac{1}{3} \sum_{k=n+1}^{2n} \frac{1}{k}$$

$$= \sum_{k=n+1}^{2n} \frac{1}{k} + \sum_{k=n}^{3n} \frac{1}{2k+1} - \frac{1}{3} \sum_{k=n+1}^{2n} \frac{1}{k},$$

from which (5.1) easily follows.

Corollary. $$\tfrac{1}{2} \operatorname{Log} 3 + \tfrac{1}{3} \operatorname{Log} 4 = \varphi(6).$$

Proof. Letting n tend to ∞ in (5.1) and employing (1.4), we achieve the desired equality.

Example 1. $$\tfrac{1}{4} \operatorname{Log} 2 = \sum_{k=1}^{\infty} \frac{1}{\{2(2k-1)\}^3 - 2(2k-1)}.$$

Proof. The right side above is equal to $\frac{1}{2}\{\varphi(2) - \varphi(4)\}$. Hence, the result follows from the corollaries to Entries 1 and 4.

Example 2. $$\operatorname{Log} 2 = 1 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{(2k)^3 - 2k}.$$

Proof. The right side above is $2\varphi(4) - \varphi(2)$, and so the example again follows from the corollaries to Entries 1 and 4.

Example 3. *For each positive integer n,*

$$2\varphi(4, n) = \varphi(2, 2n) + \tfrac{1}{2}\varphi(2, n) + \frac{1}{(4n+1)(4n+2)}. \tag{5.2}$$

Proof. From Entries 1 and 4, respectively,

$$\varphi(2, n) = 2 \sum_{k=n+1}^{2n} \frac{1}{k} + \frac{1}{2n+1} \tag{5.3}$$

and

$$\varphi(4, n) = \sum_{k=n+1}^{2n} \frac{1}{k} + \sum_{k=n}^{2n} \frac{1}{2k+1}. \tag{5.4}$$

Thus, the right side of (5.2) may be written as

$$2 \sum_{k=2n+1}^{4n} \frac{1}{k} + \sum_{k=n+1}^{2n} \frac{1}{k} + \frac{2}{4n+1} = 2 \sum_{k=n+1}^{2n} \frac{1}{k} + 2 \sum_{k=n}^{2n} \frac{1}{2k+1} = 2\varphi(4, n),$$

which completes the proof.

Example 4. *For each positive integer n,*

$$\varphi(4, n) = \frac{1}{2} \sum_{k=n+1}^{2n} \frac{1}{k} + \sum_{k=2n+1}^{4n+1} \frac{1}{k}.$$

Proof. This expression for $\varphi(4, n)$ arises from a rearrangement of the terms in (5.4).

Example 5. $\frac{1}{4} \operatorname{Log} 3 - \frac{1}{3} \operatorname{Log} 2 = \sum_{k=1}^{\infty} \frac{1}{\{3(2k-1)\}^3 - 3(2k-1)}$.

Proof. The right side above is $\frac{1}{2}\{\varphi(3) - \varphi(6)\}$, and so the result follows immediately from the corollaries of Entries 2 and 5.

Example 6.
$$\frac{4}{3} \operatorname{Log} 2 = 1 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{(3k)^3 - 3k}.$$

Proof. The right side above is $2\varphi(6) - \varphi(3)$, and so the result is a consequence of the corollaries to Entries 2 and 5.

Example 7. *For each positive integer n,*

$$2\varphi(6, n) + \tfrac{1}{3}\varphi(2, n) = \varphi(3, n) + \varphi(2, 3n) + \frac{2}{(6n+1)(6n+2)(6n+3)}. \tag{5.5}$$

Proof. By Entries 2 and 5, respectively,

$$\varphi(3, n) = \sum_{k=n+1}^{3n+1} \frac{1}{k} \tag{5.6}$$

and

$$\varphi(6, n) = \frac{2}{3} \sum_{k=n+1}^{2n} \frac{1}{k} + \sum_{k=n}^{3n} \frac{1}{2k+1}. \tag{5.7}$$

Thus, by (5.3) and (5.7), the left side of (5.5) is equal to

$$\begin{aligned} 2 \sum_{k=n+1}^{2n} \frac{1}{k} + 2 \sum_{k=n}^{3n} \frac{1}{2k+1} + \frac{1}{3(2n+1)} &= 2 \sum_{k=n+1}^{6n+1} \frac{1}{k} - \sum_{k=n+1}^{3n} \frac{1}{k} + \frac{1}{6n+3} \\ &= \sum_{k=n+1}^{3n} \frac{1}{k} + 2 \sum_{k=3n+1}^{6n+1} \frac{1}{k} + \frac{1}{6n+3} \\ &= \sum_{k=n+1}^{3n+1} \frac{1}{k} + 2 \sum_{k=3n+1}^{6n} \frac{1}{k} \\ &\quad + \frac{36n^2 + 30n + 8}{(6n+1)(6n+2)(6n+3)}, \end{aligned} \tag{5.8}$$

By (5.3) and (5.6), the far right side of (5.8) is easily seen to be equal to the right side of (5.5).

Example 8.

$$\sum_{k=2}^{13} A\left(\frac{1}{k}\right) = \frac{\pi}{2} + 2A\left(\frac{1}{4}\right) + A\left(\frac{2}{49}\right) + A\left(\frac{3}{232}\right) + A\left(\frac{4}{715}\right).$$

Proof. Apply (3.1) with $n = 1$ and with $n = 4$. Adding the two results yields the desired equality.

Example 9. *For each positive integer n,*

$$2\sum_{k=1}^{n+1} A\left(\frac{1}{n+k}\right) = A\left(\frac{n+1}{n}\right) + \sum_{k=1}^{n} A\left(\frac{2k}{8k^4+2k^2+1}\right) + 2\sum_{k=1}^{n} A\left(\frac{1}{k(4k^2+3)}\right). \tag{5.9}$$

Proof. Rewriting the left side of (5.9) and then employing (0.1) and (0.2) several times, we find that

$$\begin{aligned}
&2\sum_{k=1}^{n+1} A\left(\frac{1}{n+k}\right)\\
&= A(1) + A\left(\frac{1}{2n+1}\right) + 2\sum_{k=1}^{n}\left\{2A\left(\frac{1}{2k}\right) - A\left(\frac{1}{k}\right)\right\}\\
&\quad + \sum_{k=1}^{n}\left\{\left[A\left(\frac{1}{2k-1}\right) - A\left(\frac{1}{2k}\right)\right] - \left[A\left(\frac{1}{2k}\right) - A\left(\frac{1}{2k+1}\right)\right]\right\}\\
&= A\left(\frac{n+1}{n}\right) + 2\sum_{k=1}^{n}\left\{A\left(\frac{4k}{4k^2-1}\right) - A\left(\frac{1}{k}\right)\right\}\\
&\quad + \sum_{k=1}^{n}\left\{A\left(\frac{1}{4k^2-2k+1}\right) - A\left(\frac{1}{4k^2+2k+1}\right)\right\}\\
&= A\left(\frac{n+1}{n}\right) + 2\sum_{k=1}^{n} A\left(\frac{1}{k(4k^2+3)}\right) + \sum_{k=1}^{n} A\left(\frac{2k}{8k^4+2k^2+1}\right).
\end{aligned}$$

Example 10. *For each positive integer n,*

$$\begin{aligned}
&\sum_{k=1}^{n} A\left(\frac{1}{n+k}\right) + \sum_{k=0}^{n} A\left(\frac{1}{2n+2k+1}\right)\\
&= \frac{\pi}{4} + \sum_{k=1}^{n} A\left(\frac{9k}{32k^4+22k^2-1}\right) + \sum_{k=1}^{n} A\left(\frac{4k}{128k^4+8k^2+1}\right).
\end{aligned} \tag{5.10}$$

Proof. We first rewrite the left side of (5.10) and then use (0.1) and (0.2) several times. Accordingly, we get

$$\sum_{k=1}^{n} A\left(\frac{1}{n+k}\right)+\sum_{k=0}^{n} A\left(\frac{1}{2n+2k+1}\right)$$
$$=A(1)+\sum_{k=1}^{n}\left\{2A\left(\frac{1}{4k}\right)-\left[A\left(\frac{1}{k}\right)-A\left(\frac{1}{2k}\right)\right]\right\}$$
$$+\sum_{k=1}^{n}\left\{\left[A\left(\frac{1}{4k-1}\right)-A\left(\frac{1}{4k}\right)\right]-\left[A\left(\frac{1}{4k}\right)-A\left(\frac{1}{4k+1}\right)\right]\right\}$$
$$=\frac{\pi}{4}+\sum_{k=1}^{n}\left\{A\left(\frac{8k}{16k^2-1}\right)-A\left(\frac{k}{2k^2+1}\right)\right\}$$
$$+\sum_{k=1}^{n}\left\{A\left(\frac{1}{16k^2-4k+1}\right)-A\left(\frac{1}{16k^2+4k+1}\right)\right\}$$
$$=\frac{\pi}{4}+\sum_{k=1}^{n} A\left(\frac{9k}{32k^4+22k^2-1}\right)+\sum_{k=1}^{n} A\left(\frac{4k}{128k^4+8k^2+1}\right).$$

Entry 6. *Let k and n be nonnegative integers and define* $A_k = 3^k(n+\frac{1}{2})-\frac{1}{2}$. *Then if r is a positive integer,*

$$\sum_{k=n+1}^{A_r}\frac{1}{k}=r+2\sum_{k=0}^{r-1}(r-k)\sum_{j=A_{k-1}+1}^{A_k}\frac{1}{(3j)^3-3j},$$

where we define $A_{-1}=0$.

Proof. This proof was given by Ramanujan. First, it is easily shown that $A_{k+1}=3A_k+1$, $k\geq 0$. Hence, by Entry 2, with $n=A_k$,

$$\sum_{j=A_k+1}^{A_{k+1}}\frac{1}{j}=\varphi(3, A_k).$$

Now sum both sides of this equality on k, $0\leq k\leq r-1$, to obtain

$$\sum_{j=n+1}^{A_r}\frac{1}{j}=r+2\sum_{k=0}^{r-1}\sum_{j=1}^{A_k}\frac{1}{(3j)^3-3j}.$$

Rearranging the right side above, we deduce the desired equality.

Corollary. *For each positive integer r,*

$$\sum_{k=1}^{(3^r-1)/2}\frac{1}{k}=r+2\sum_{k=1}^{r-1}(r-k)\sum_{j=A_{k-1}+1}^{A_k}\frac{1}{(3j)^3-3j},$$

where $A_k=(3^k-1)/2$, $k\geq 0$.

Proof. This corollary is the case $n=0$ of Entry 6.

At this point, Ramanujan claims that if a,, . . . , a, are in arithmetic progression and if a, and a, are large then $\sum_{k=1}^{n} 1/a_k$ is approximately equal to $2n/(a_1 +$ a,). Unfortunately, this remark is false. For example, consider $S_n = \sum_{k=n+1}^{3n+1} 1/k$. If Ramanujan were correct, then for large n, S_n would approximately be equal to $2(2n+1)/(4n+2) = 1$. However, by (2.2), S_n tends to Log 3, as n tends to ∞.

Nonetheless, Ramanujan's assertion is correct if n/a_1 is "small," as we now demonstrate. Letting a, d, and n denote positive integers, define

$$S(a, d, n) = \sum_{k=0}^{n} \frac{1}{a+kd}.$$

Now, $2n/(a_1 +$ a,) in Ramanujan's notation becomes $2(n+1)/(2a+nd)$ in our notation. Thus, we wish to examine

$$S(a, d, n) - \frac{2(n+1)}{2a+nd} = \sum_{k=0}^{n} \frac{(n-2k)d}{(a+kd)(2a+nd)}$$

$$= \frac{d}{2a+nd}\left\{n\left(\frac{1}{a} - \frac{1}{a+nd}\right)\right.$$

$$+ (n-2)\left(\frac{1}{a+d} - \frac{1}{a+(n-1)d}\right)$$

$$\left. + (n-4)\left(\frac{1}{a+2d} - \frac{1}{a+(n-2)d}\right) + \cdots + T_n\right\},$$

where

$$T_n = \begin{cases} \dfrac{1}{a+(n-1)d/2} - \dfrac{1}{a+(n+1)d/2}, & \text{if } n \text{ is odd,} \\[2ex] \dfrac{2}{a+(n-2)d/2} - \dfrac{2}{a+(n+2)d/2}, & \text{if } n \text{ is even.} \end{cases}$$

Hence, with d fixed,

$$S(a, d, n) - \frac{2(n+1)}{2a+nd} = O\left\{\frac{n^2}{a}\left(\frac{1}{a} - \frac{1}{a+nd}\right)\right\} = O\left\{\left(\frac{n}{a}\right)^3\right\},$$

as n/a tends to 0. Thus, under this assumption, Ramanujan's approximation is, indeed, valid.

Example 1. $$\sum_{k=1}^{13} \frac{1}{k} = 3 + \frac{1}{6} + \frac{1}{105} + \frac{1}{360} + \frac{1}{858}.$$

Proof. In the previous corollary, let $r = 3$. Since A, = 0, A, = 1, $A_2 = 4$, and **A,** = 13, we find that

$$\sum_{k=1}^{13} \frac{1}{k} = 3 + \frac{4}{3^3 - 3} + 2\left(\frac{1}{6^3 - 6} + \frac{1}{9^3 - 9} + \frac{1}{(12)^3 - 12}\right),$$

and the result follows.

Example 2. $$H \equiv \sum_{k=1}^{1000} \frac{1}{k} = 7\tfrac{1}{2} \quad \text{“very nearly.”}$$

Proof. In the previous corollary, let $r = 7$. In addition to **A,, A,, A,,** and **A,** calculated above, we need the values **A,** = 40, **A,** = 121, **A,** = 364, and **A,** = 1093. Thus,

$$\begin{aligned}\sum_{k=1}^{1093} \frac{1}{k} &= 7 + \tfrac{1}{2} + 5\left(\frac{1}{105} + \frac{1}{360} + \frac{1}{858}\right) \\ &\quad + 8 \sum_{k=5}^{13} \frac{1}{(3k)^3 - 3k} + 6 \sum_{k=14}^{40} \frac{1}{(3k)^3 - 3k} \\ &\quad + 4 \sum_{k=41}^{121} \frac{1}{(3k)^3 - 3k} + 2 \sum_{k=122}^{364} \frac{1}{(3k)^3 - 3k} \\ &= 7 + 0.5 + 0.067335442\ldots + 0.006435448\ldots \\ &\quad + 0.000541282\ldots + 0.000040137\ldots + 0.000002230\ldots \\ &= 7.574354539\ldots \end{aligned} \tag{6.1}$$

Next, by the remarks prior to Example 1,

$$\sum_{k=1001}^{1093} \frac{1}{k} \approx \frac{186}{2094} = 0.088825214\ldots. \tag{6.2}$$

Thus, from (6.1) and (6.2) we conclude that $H \approx 7.48552932\ldots$. This is probably the method that Ramanujan employed to estimate H. On the other hand, by using the Euler-Maclaurin summation formula or a programable calculator, it can readily be shown that $H = 7.48547086\ldots$ In any event, the estimate of $7\frac{1}{2}$ for H is not as good as Ramanujan would lead us to believe.

Entry 7. ***Let $n > 0$ and suppose that r is a natural number. Then***

$$\sum_{k=0}^{r-1} A\left(\frac{2}{(n + 2k + 1)^2}\right) = A\left(\frac{2r}{n^2 + 2nr + 1}\right).$$

Proof. The proof is very briefly sketched by Ramanujan. Since $n > 0$, it follows from (0.2) that

$$A\left(\frac{1}{n + 2k}\right) - A\left(\frac{1}{n + 2k + 2}\right) = A\left(\frac{2}{(n + 2k + 1)^2}\right). \tag{7.1}$$

Now sum both sides of (7.1) on k, $0 \le k \le r - 1$, to get

$$A\left(\frac{1}{n}\right) - A\left(\frac{1}{n+2r}\right) = \sum_{k=0}^{r-1} A\left(\frac{2}{(n+2k+1)^2}\right). \tag{7.2}$$

An application of (0.2) on the left side of (7.2) completes the proof.

Corollary. *For $n > 0$,*

$$\sum_{k=0}^{\infty} A\left(\frac{2}{(n+2k+1)^2}\right) = A\left(\frac{1}{n}\right). \tag{7.3}$$

Proof. Let r tend to ∞ in (7.2).

Example 1. *For $n > 0$,*

$$\sum_{k=1}^{\infty} A\left(\frac{2}{(n+k)^2}\right) = A\left(\frac{2n+1}{n^2+n-1}\right) + \rho(n), \tag{7.4}$$

where $\rho(n) = \pi$ if $n < (\sqrt{5}-1)/2$ and $\rho(n) = 0$ otherwise.

Proof. The proof is sketched by Ramanujan. From the previous corollary and (0.1),

$$\begin{aligned}\sum_{k=1}^{\infty} A\left(\frac{2}{(n+k)^2}\right) &= \sum_{k=0}^{\infty} A\left(\frac{2}{(n+2k+1)^2}\right) + \sum_{k=0}^{\infty} A\left(\frac{2}{(n+2k+2)^2}\right)\\ &= A\left(\frac{1}{n}\right) + A\left(\frac{1}{n+1}\right)\\ &= A\left(\frac{2n+1}{n^2+n-1}\right) + \rho(n),\end{aligned}$$

since $\rho(n) = \pi$ if and only if $n^2 + n < 1$, i.e., $n < (\sqrt{5}-1)/2$. This completes the proof.

Example 2. *For $n > 0$,*

$$\sum_{k=1}^{\infty} (-1)^{k+1} A\left(\frac{2}{(n+k)^2}\right) = A\left(\frac{1}{n^2+n+1}\right). \tag{7.5}$$

Proof. The proof is very similar to that of Example 1.

Example 3. *For $n > 0$,*

$$\sum_{k=1}^{\infty} A\left(\frac{1}{2(n+k)^2}\right) = A\left(\frac{1}{2n+1}\right). \tag{7.6}$$

Proof. Replace n by $2n + 1$ in the corollary to Entry 7.

Example 4.

$$\sum_{k=1}^{\infty} A\left(\frac{2}{k^2}\right) = \frac{3\pi}{4}.$$

Proof. Since the series on the left side of (7.4) converges uniformly on $0 \le n \le 1$, we may let n tend to 0 on both sides of (7.4). The desired result then immediately follows.

Example 4 and the first equality in Example 5 which follows were apparently first established by Glaisher [2] in 1878. This paper contains many other examples of this sort. Example 4 is a problem in Chrystal's book [1, p. 357] as well as in Loney's textbook [1, Part II, p. 206]. The latter fact is interesting because the borrowing of Loney's book from a friend while in fourth form was evidently a pivotal event in Ramanujan's mathematical development [15, p. xii]. Still another proof of Example 4 can be found in Wheelon's book [1, p. 46].

Example 5.

$$\sum_{k=1}^{\infty} A\left(\frac{1}{2k^2}\right) = \frac{\pi}{4} = \sum_{k=1}^{\infty} (-1)^{k+1} A\left(\frac{2}{k^2}\right).$$

Proof. The series on the left sides of (7.5) and (7.6) each converge uniformly on $0 \le n \le 1$. Letting n tend to 0 in (7.5) and (7.6), we immediately deduce the evaluations above.

Example 6.

$$\sum_{k=1}^{\infty} A\left(\frac{1}{(1+\sqrt{2}\,k)^2}\right) = \frac{\pi}{8}$$

Proof. In Example 3 let $n = 1/\sqrt{2}$. A short calculation shows that $\tan(\pi/8) = 1/(\sqrt{2}+1)$, and so the result readily follows.

Example 7.

$$\sum_{k=1}^{\infty} A\left(\frac{8}{(2k-1+\sqrt{5})^2}\right) = \frac{\pi}{2}.$$

Proof. Since the left side of (7.4) converges uniformly on $0 \le n \le 1$, we may let n tend to $(\sqrt{5}-1)/2 + 0$. The sought equality then follows.

Example 8.

$$\sum_{k=0}^{\infty} A\left(\frac{2}{(2k+1)^2}\right) = \frac{\pi}{2}.$$

Proof. The series on the left side of (7.3) converges uniformly on $0 \le n \le 1$. Letting n tend to 0 in (7.3) yields the desired result.

Example 8 is also found in Glaisher's paper [2].

In Entry 8 Ramanujan considers an entire function f(z) with zeros z_1, z_2, ... He evidently assumes that $\sum_{k=1}^{\infty} 1/|z_k|$ converges and states a corresponding special case of the Hadamard factorization theorem. (See, for example, Titchmarsh's book [1, p. 250].) He then takes the logarithmic derivative of this product representation for f(z) and evaluates it at $z = 0$.

In Entries 9(i) and (ii) the familiar product representations for sin x and cos x are stated. Corollaries 1 and 2 of Entry 9 give the well-known product representations of sinh x and cosh x.

Corollary 3. *For each complex number x,*

$$\cos\left(\frac{x}{4}\right) + \sin\left(\frac{x}{4}\right) = \prod_{k=0}^{\infty}\left(1 + \frac{(-1)^k x}{(2k+1)\pi}\right).$$

Corollary 4. *Let x and a be complex, where a is not an integral multiple of π. Then*

$$\frac{\sin(x+a)}{\sin a} = \frac{x+a}{a}\prod_{k=1}^{\infty}\left\{\left(1 - \frac{x}{k\pi - a}\right)\left(1 + \frac{x}{k\pi + a}\right)\right\}.$$

Corollary 3 is easily derived from Corollary 4 by setting $a = \pi/4$ and replacing x by x/4. Corollaries 3 and 4 are rather straightforward exercises which can be found in Bromwich's book [1, p. 224], for example, and so it is pointless to give a proof of Corollary 4 here.

Example 1. *Let x and a be complex, where a is not an odd multiple of $\pi/2$. Then*

$$\frac{\cos(x+a)}{\cos a} = \prod_{k=1}^{\infty}\left\{\left(1 - \frac{x}{(k-\frac{1}{2})\pi - a}\right)\left(1 + \frac{x}{(k-\frac{1}{2})\pi + a}\right)\right\}.$$

Example 1 is easily derived from Corollary 4 by replacing a by $a + \pi/2$. Example 2 below follows from Corollary 4 and Example 1 upon the use of the identity

$$1 + \frac{\sin x}{\sin a} = \frac{\sin\{\frac{1}{2}(x+a)\}\cos\{\frac{1}{2}(x-a)\}}{\sin(\frac{1}{2}a)\qquad\cos(\frac{1}{2}a)}.$$

Example 2. *Let x and a be complex, where a is not an integral multiple of π. Then*

$$1 + \frac{\sin x}{\sin a} = \frac{a+x}{a}\prod_{k=1}^{\infty}\left\{\left(1 - \frac{x}{2k\pi - a}\right)\left(1 + \frac{x}{2k\pi + a}\right)\right.$$
$$\left.\times\left(1 - \frac{x}{(2k-1)\pi + a}\right)\left(1 + \frac{x}{(2k-1)\pi - a}\right)\right\}.$$

Next, Ramanujan asserts that if the value of $F(x) = \prod_{k=1}^{\infty}(1 + a_k x)$ is known, then it is possible to find the value of $\prod_{k=1}^{\infty}(1 + a_k^n x^n)$, where n is a positive integer. Ramanujan's declaration evidently arises from the identity

$$\prod_{k=1}^{\infty} (1 + a_k^n x^n) = \prod_{j=1}^{n} F(-\omega^{2j-1} x),$$

which is a consequence of the factorization

$$1 + a^n x^n = \prod_{j=1}^{n} (1 - a\omega^{2j-1} x),$$

where $\omega = \exp(\pi i/n)$.

In Entry 10, the familiar partial fraction decompositions of cot x, tan x, csc x, and sec x are given. (See, for example, Bromwich's text [1, pp. 217, 225].)

Entry 11. Let *x and a be real. Then*

$$A\left(\frac{x}{a}\right) + \sum_{k=1}^{\infty} \{ A (\&) - A (\&) \} = A(\tanh x \cot a). \quad (11.1)$$

Proof. The main idea for the proof is indicated by Ramanujan. By Corollary 4 of Entry 9,

$$\operatorname{Im} \operatorname{Log}\left(\frac{\sin(a+ix)}{\sin a}\right) = \operatorname{Im} \operatorname{Log}\left\{\left(1+\frac{ix}{a}\right)\prod_{k=1}^{\infty}\left(1-\frac{ix}{k\pi-a}\right)\left(1+\frac{ix}{k\pi+a}\right)\right\}$$

$$= A\left(\frac{x}{a}\right) + \sum_{k=1}^{\infty}\left\{A\left(\frac{x}{k\pi+a}\right) - A\left(\frac{x}{k\pi-a}\right)\right\}, \quad (11.2)$$

up to an additive multiple of π. On the other hand,

$$\operatorname{Im} \operatorname{Log}\left(\frac{\sin(a+ix)}{\sin a}\right) = \operatorname{Im} \operatorname{Log}(\cosh x + i \sinh x \cot a)$$

$$= A(\tanh x \cot a), \quad (11.3)$$

up to an additive multiple of π. Combining (11.2) and (11.3), we have shown that (11.1) is valid up to an additive multiple of π. We now show that this additive multiple of π is, indeed, 0.

First, if a is a multiple of π, it is readily checked that (11.1) is valid. Suppose now that a is fixed but not a multiple of π. For x = 0, (11.1) is certainly true. Since both sides of (11.1) are continuous functions of x, that additive multiple of π must be 0 for all x. Since a is arbitrary, the proof is complete.

Corollary 1. *Let x and a be real. Then*

$$\sum_{k=-\infty}^{\infty} (-1)^k A\left(\frac{x}{k\pi+a}\right) = A(\sinh x \csc a). \quad (11.4)$$

Proof. First, by Example 2 of Entry 9,

$$\begin{aligned}A(\sinh x \csc a) &= \operatorname{Im} \operatorname{Log}\left(1 + i\frac{\sinh x}{\sin a}\right)\\ &= \operatorname{Im} \operatorname{Log}\left\{\left(1+\frac{ix}{a}\right)\prod_{k=1}^{\infty}\left(1-\frac{ix}{2k\pi - a}\right)\left(1+\frac{ix}{2k\pi+a}\right)\right.\\ &\quad \times\left.\left(1-\frac{ix}{(2k-1)\pi+a}\right)\left(1+\frac{ix}{(2k-1)\pi-a}\right)\right\}\\ &= \sum_{k=-\infty}^{\infty}(-1)^k A\left(\frac{x}{k\pi+a}\right),\end{aligned}$$

up to an additive multiple of π. Thus, (11.4) is valid up to an additive multiple of π. To show that this multiple of π is 0, we proceed in the same manner as in the proof of Entry 11.

Corollary 2. *For real x,*

$$\sum_{k=0}^{\infty}(-1)^k A\left(\frac{x}{2k+1}\right) = A\left(\tanh\left(\frac{\pi x}{4}\right)\right).$$

Proof. Replacing x by $\pi x/4$ and setting $a = \pi/4$ in Entry 11, we readily achieve the desired formula.

Corollary 3. *For real x,*

$$\sum_{k=-\infty}^{\infty}(-1)^k A\left(\frac{x}{4k+1}\right) = A\left(\sqrt{2}\sinh\left(\frac{\pi x}{4}\right)\right).$$

Proof. In Corollary 1 of Entry 11 replace x by $\pi x/4$ and let $a = \pi/4$ to deduce the sought formula.

The next two examples are obtained by replacing a by $\pi/2 - a$ in Entry 11 and Corollary 1, respectively. In the second notebook there is a minor misprint in Example 1.

Example 1. *For x and a real,*

$$\sum_{k=1}^{\infty}\left\{A\left(\frac{x}{\frac{1}{2}(2k-1)\pi - a}\right) - A\left(\frac{x}{\frac{1}{2}(2k-1)\pi + a}\right)\right\} = A(\tanh x \tan a).$$

Example 2. *For x and a real,*

$$\sum_{k=-\infty}^{\infty}(-1)^k A\left(\frac{x}{\frac{1}{2}(2k+1)\pi - a}\right) = A(\sinh x \sec a).$$

Entry 11 is an exercise in both the books of Chrystal [1, p. 373] and Loney

[1, Part II, p. 2083. Corollary 2 is also a problem in Loney's book [1]. Entry 11, Corollaries 1 and 2, and Example 1 are given in Hansen's tables [1, p. 276]. Several other arctangent series in the spirit of those given above are summed in this compendium. Ramanujan himself summed other arctangent series in [10], [15, p. 42]. Glasser and Klamkin [1] have summed several arctangent series in an elementary fashion. Further examples of arctangent series are found in Bromwich's book [1, pp. 314–315].

Example 3.
$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^3}\right)=\frac{1}{\pi}\cosh\left(\frac{\pi\sqrt{3}}{2}\right).$$

Example 4.
$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^3}\right)=\frac{1}{3\pi}\cosh\left(\frac{\pi\sqrt{3}}{2}\right)$$

Examples 3 and 4 constitute the second problem that Ramanujan submitted to the *Journal of the Zndian Mathematical Society* [2], [15, p. 322]. In a later paper [12], [15, pp. 50–52], Ramanujan studied the more general product

$$\prod_{k=0}^{\infty}\left\{1+\left(\frac{x}{a+kd}\right)^3\right\}.$$

In Entry 12, Ramanujan presents a method for approximating the root z_0 of smallest modulus of the equation

$$\sum_{k=1}^{\infty} A_k z^k = 1. \tag{12.1}$$

It is assumed that all other roots of (12.1) have moduli strictly greater than z_0 |. For z sufficiently small, write

$$\frac{1}{1-\sum_{k=1}^{\infty} A_k z^k}=\sum_{k=1}^{\infty} P_k z^{k-1}$$

It follows easily that $P_1 = 1$ and

$$P_n=\sum_{j=1}^{n-1} A_j P_{n-j}, \qquad n \geq 2. \tag{12.2}$$

Now assume that lim,, $_\infty$ P_n/P_{n+1} exists and is equal to L. Then, of course, the radius of convergence of $\sum_{k=1}^{\infty} P_k z^k$ is equal to L|. Moreover, by a theorem of Fabry in the book of Hadamard and Mandelbrojt [1, pp. 39–40], L is a singularity of the function represented by this series. It follows that if the radius of convergence of the series $\sum_{k=1}^{\infty} A_k z^k$ is greater than L |, then z = *Lis* a root of (12.1). Ramanujan's discourse is characteristically brief; he gives (12.2) and claims, with no hypotheses, that P_n/P_{n+1} approaches a root of (12.1).

In the case that (12.1) is an algebraic equation, this method is originally due to Daniel Bernoulli. Accounts of Bernoulli's method may be found in the books of Whittaker and Robinson [1, pp. 98–99] and Henrici [1, p. 663]. Usually, a change of variable is made so that the method yields the approximate value of the root with largest modulus. Bernoulli's method has been generalized by Aitken [1] who found a way to approximate any root of a polynomial.

Ramanujan concludes this chapter by giving six examples to illustrate his method. He takes $P_0 = 0$, and so the first convergent is always 0/1.

Example 1. *The roots of* $x+x^2 = 1$ *are* $(-1 \pm \sqrt{5})/2$, *and so* $(\sqrt{5}-1)/2 = 0.618034\ldots$ *is the root of least modulus. Ramanujan gives the jirst eight convergents to this root with the last being* $P_7/P_8 = 13/21 = 0.619048\ldots$.

Example 2. *By Newton's method the real root of* $x+x^2+x^3 = 1$ *is* 0.543689013.. . . *Ramanujan gives the jirst eight convergents to this root with the last equal to* 24/44 = *0.5454. . . .*

Example 3. *Ramanujan lists the jirst ten convergents to the real root of* $x + x^3 = 1$, *with the last convergent being* 13/19 = 0.684210... . *By Newton's method, this root is 0.682327804.. .*

Example 4. *The last polynomial equation examined by Ramanujan is* $2x + x^2 + x^3 = 1$. *He calculates seven convergents to the real root andfinds the seventh to be* 84/214 = 0.392523.... *This root is* 0.392646782..., *by Newton's method.*

At this point, Ramanujan claims that "If p/q and r/s are two consecutive convergents to x, then we may take $(mp + nr)/(mq + ns)$ in a suitable manner equivalent to x." If m and n are to be taken as real, then Ramanujan's remark is pointless, for then this ratio may be made to take any real value. On the other hand, if m and n are to be understood as positive, then Ramanujan's assertion is false. Ramanujan's claim would be valid if the limit L were always between two consecutive convergents. However, this may not be true. For example, the last three convergents 13/33, 33/84, and 84/214 given by Ramanujan in Example 4 satisfy the inequalities $13/33 > 33/84 > 84/214$.

Example 1. *In this example Ramanujan examines* $e^x = 2$ *and finds the jirst six convergents to* Log *2* = 0.69315.. . . *The sixth convergent is* 375/541 = 0.69316.. . .

Example *2. In this last example Ramanujan approximates the root of* $e^{-x} = x$. *He calculates five convergents with the last one equal to* 148/261 = 0.567049. . . . *By Newton's method, the root is* 0.567143290.. . .

E. M. Wright has written several papers [1]–[5], in which he has studied solutions of equations generalizing the one in the last example. Such equations are very important in the theory of differential-difference equations.

CHAPTER 3

Combinatorial Analysis and Series Inversions

Although no combinatorial problems are mentioned in Chapter 3, much of the content of this chapter belongs under the umbrella of combinatorial analysis. Another primary theme in Chapter 3 revolves around series expansions of various types. However, the deepest and most interesting result in Chapter 3 is Entry 10, which separates the two main themes but which has some connections with the former. Entry 10 offers a highly general and potentially very useful asymptotic expansion for a large class of power series. As with Chapter 2, Ramanujan very briefly sketches the proofs of some of his findings, including Entry 10.

Some of the results of Chapter 3 can be traced back to Lambert, Lagrange, Euler, Rothe, Abel, and others. On the other hand, much of Ramanujan's work in Chapter 3 has been rediscovered by others unaware of his work. For example, the single variable Bell polynomials were first thoroughly examined in print by J. Touchard [1] in 1933 and by E. T. Bell [1] in 1934, but Ramanujan had already discovered many properties of these polynomials in Chapter 3. Also, several other results were rediscovered and considerably generalized by H. W. Gould [1]–[6] in the late 1950's and early 1960's.

The first nine sections of Chapter 3 comprise a total of 45 formulas. The majority of these results involve properties of the Bell numbers and single variable Bell polynomials and are not very difficult to establish.

Entry 10 is enormously interesting and is certainly the most difficult result to prove in this chapter. Ramanujan proposes an asymptotic expansion for a wide class of power series and provides a sketch of his proof. His argument, however, is formal and not mathematically rigorous. He then gives three very intriguing applications of this theorem. Unfortunately, for none of these applications are the hypotheses, implied in his formal argument, satisfied. We shall establish Ramanujan's asymptotic formula under much weaker assumptions than those implied by his argument. Ramanujan's three examples are then seen to be special cases of our theorem. As is to be expected, our method of attack is much different from that of Ramanujan, but since his argument is interesting, we shall provide a sketch of it.

The content of Sections 11-17 is not unrelated to that of Sections 1-9. However, the proofs are somewhat more formidable. The key problem is to express powers of x by certain series, where x is a root of a particular equation. This theme appears to have commenced in the work of Lambert [1], Lagrange [1], and Euler [4] and has had a fairly long history. These expansions can be derived via the Lagrange inversion theorem. This theorem is found in Carr's book [1], and Ramanujan's quarterly reports bear testimony that he was well acquainted with Lagrange's theorem. However, as the quarterly reports further indicate, Ramanujan possessed another technique, indeed, a very ingenious, novel one, for deriving these expansions. Entry 13 is central in Ramanujan's theory and is the ground for several variations in the sequel. Example 1 of Entry 15 is an extremely interesting result. Entries 16 and 17 do not seem to have been expanded upon in the literature and would appear to be a basis for further fruitful research.

Entry 1. *Let $f(z)$ be analytic on $|z| < R_1$, where $R_1 > 1$, and let $g(z) = \sum_{k=0}^{\infty} Q_k z^k$ be analytic on $|z| < R_2$, where $R_2 > 0$. Define P_k, $0 \le k < \infty$, by $\sum_{k=0}^{\infty} P_k z^k = e^z g(z)$, where $|z| < R_2$. Suppose that $\sum_{j=0}^{\infty} Q_j \sum_{k=0}^{\infty} f^{(j+k)}(0)/k!$ converges and that this repeated summation may be replaced by a summation along diagonals, i.e., $j + k = n$, $0 \le n < \infty$. Then*

$$\sum_{n=0}^{\infty} P_n f^{(n)}(0) = \sum_{n=0}^{\infty} Q_n f^{(n)}(1).$$

Proof. Since $R_1 > 1$, we find from Taylor's theorem that

$$f^{(j)}(1) = \sum_{k=0}^{\infty} \frac{f^{(j+k)}(0)}{k!}, \qquad 0 \le j < \infty.$$

Hence,

$$\sum_{j=0}^{\infty} Q_j f^{(j)}(1) = \sum_{j=0}^{\infty} Q_j \sum_{k=0}^{\infty} \frac{f^{(j+k)}(0)}{k!} = \sum_{n=0}^{\infty} P_n f^{(n)}(0),$$

which can readily be seen from the definition of P_n.

Corollary 1. *Suppose that the hypotheses of Entry 1 are satisfied for $f(z) = (1 + xz)^n$, where $|x| < 1$ and n is arbitrary. Then*

$$\sum_{k=0}^{\infty} P_k \frac{x^k}{\Gamma(n-k+1)} = \sum_{k=0}^{\infty} Q_k \frac{x^k(1+x)^{n-k}}{\Gamma(n-k+1)}.$$

Proof. Elementary calculations yield $f^{(k)}(0) = \Gamma(n+1)x^k/\Gamma(n-k+1)$ and $f^{(k)}(1) = \Gamma(n+1)x^k(1+x)^{n-k}/\Gamma(n-k+1)$, $0 \le k < \infty$. The desired equality now follows.

Corollary 2 is simply an alternative formulation of Entry 1, and so we shall not bother to state this corollary.

Note that the next entry gives concrete examples for P_k and Q_k in Entry 1. Ramanujan indicates two proofs of Entry 2. The first is purely formal, while the second is more easily made rigorous.

Entry 2. ***For all complex x and z, define***

$$\varphi(z) = \sum_{k=1}^{\infty} \frac{x^k}{(z+k-1)(k-1)!}.$$

Then

$$\varphi(z) = e^x \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{z(z+1)(z+2)\ldots(z+k-1)}. \tag{2.1}$$

First proof. By employing the Maclaurin series for e^x and integrating termwise, Ramanujan gets

$$\begin{aligned}\varphi(z) &= \frac{1}{x^{z-1}} \sum_{k=0}^{\infty} \frac{x^{z+k}}{(z+k)k!} = \frac{1}{x^{z-1}} \int x^{z-1} e^x \, dx \\ &= e^x \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{z(z+1)(z+2)\ldots(z+k-1)},\end{aligned}$$

upon infinitely many integrations by parts.

Second proof. An easy calculation gives $z\varphi(z) + x\varphi(z+1) = xe^x$. By employing this recursion formula n times, we obtain

$$\varphi(z) - e^x \sum_{k=1}^{n} \frac{(-1)^{k-1} x^k}{z(z+1)(z+2)\ldots(z+k-1)} = \frac{(-1)^n x^n \varphi(z+n)}{z(z+1)(z+2)\ldots(z+n-1)}. \tag{2.2}$$

From the definition of $\varphi(z)$ and Stirling's formula (I6), it is not hard to see that the right side of (2.2) tends to 0 as n tends to ∞.

Corollary 1. ***Let f satisfy the hypotheses of Entry 1. Then for all z,***

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{(z+k)k!} = \sum_{k=0}^{\infty} \frac{(-1)^k f^{(k)}(1)}{z(z+1)\ldots(z+k)}.$$

Proof. Use the functions of Entry 2 in Entry 1, and the desired result immediately follows.

Corollary 2. ***For each complex number x, we have***

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \frac{x^k}{k!} = e^x \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k!\, k}.$$

Proof. In (2.1) replace x by $-x$ and z by z + 1 to get

$$\sum_{k=1}^{\infty} \frac{x^k}{(z+1)(z+2)\ldots(z+k)} = e^x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{(z+k)(k-1)!}. \tag{2.3}$$

Now differentiate both sides of (2.3) with respect to z and then set z = 0 to achieve the desired formula.

The function

$$\psi(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{(z+1)(z+2)\ldots(z+k)}, \qquad x \neq 0, \tag{3.1}$$

is a meromorphic function of z with simple poles at $z = -k$, $1 \leq k < \infty$, and thus has an essential singularity at ∞. For each fixed x, $\psi(z)$ is an inverse factorial series and has its abscissa of convergence equal to $-\infty$. Thus, the series also represents the function asymptotically as z tends to ∞ in the region

$$R_\varepsilon = \{z: -\pi + \varepsilon \leq \arg z \leq \pi - \varepsilon\}, \qquad \varepsilon > 0. \tag{3.2}$$

In Entry 3, Ramanujan obtains a second asymptotic expansion for $\psi(z)$ valid in R_ε. To describe this expansion, first define, after Ramanujan,

$$f_{-1}(x) \equiv 1, \qquad e^x f_n(x) = \sum_{k=1}^{\infty} \frac{k^n x^k}{(k-1)!}, \tag{3.3}$$

where x is any complex number and n is a nonnegative integer. In Entry 3, Ramanujan shows that as z tends to ∞ in R_ε,

$$\psi(z) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1} f_{k-1}(x)}{z^k}. \tag{3.4}$$

The series in (3.4) is divergent for all values of $x \neq 0$ and z, as can be seen directly from (3.3). Entry 3 is readily seen to be a special case of Example 1 in Section 8, and so we shall defer the proof of Entry 3 until then.

Entry 3. *Let ψ and f_n be defined by* (3.1) *and* (3.3), *respectively. Then as z tends to ∞ in R_ε, (3.4) holds.*

Entry 4. *Let a and x be arbitrary complex numbers. Then*

$$e^{x(e^a - 1)} = \sum_{n=0}^{\infty} \frac{a^n}{n!} f_{n-1}(x), \tag{4.1}$$

where $f_n(x)$ is defined by (3.3).

Proof. We have

$$e^{xe^a} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^m (ma)^n}{m!\, n!}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{x^m m^{n-1}}{(m-1)!} = \sum_{n=0}^{\infty} \frac{a^n}{n!} e^x f_{n-1}(x),$$

from which the desired result follows.

Entry 5. ***For each nonnegative integer n,***

$$f_n(x) = x \sum_{k=0}^{n} \binom{n}{k} f_{k-1}(x). \tag{5.1}$$

Proof. Differentiate both sides of (4.1) with respect to a to obtain

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} f_n(x) = xe^a e^{x(e^a - 1)} = xe^a \sum_{k=0}^{\infty} \frac{a^k}{k!} f_{k-1}(x). \tag{5.2}$$

If we now equate coefficients of a^n on the extremal sides of (5.2), we readily deduce (5.1).

It is clear from the recursion formula (5.1) that $f_n(x)$ is a polynomial of degree $n + 1$ with integral coefficients. Furthermore, for $n \geq 0, f_n(0) = 0$. Thus, following Ramanujan, we define integers $\varphi_1(n), \dots, \varphi_{n+1}(n)$, $0 \leq n < \infty$, by

$$f_n(x) = \sum_{k=1}^{n+1} \varphi_k(n) x^k. \tag{6.1}$$

The polynomials $f_n(x)$ appear to have been first systematically studied in the literature by Touchard [1] in 1933 and Bell [1] in 1934, although there is an early reference to these polynomials in Bromwich's book [1, p. 195]. They are now called single-variable Bell polynomials and are most often designated by $\varphi_n(x) = f_{n-1}(x)$, $n \geq 0$. Touchard [1], [2] and Carlitz [5] have studied these polynomials in detail and have established many arithmetic properties for them. Actually, Bell [1] introduced a much more general class of polynomials, now called Bell polynomials. In addition to Bell's papers [1], [2], extensive discussions of Bell polynomials may be found in the books of Riordan [1] and G. E. Andrews [1] which also describe combinatorial applications of Bell polynomials. The coefficients $\varphi_k(n)$ are Stirling numbers of the second kind. In the most frequent contemporary notation, $\varphi_k(n) = S(n+1, k)$. The recursion formula (5.1) is now well known as are the properties of the Stirling numbers of the second kind found in the next three entries.

Entry 6. ***Let n be a nonnegative integer and let r be a positive integer with*** $r \leq n + 1$. ***Then***

$$\sum_{k=0}^{r-1} \frac{(-1)^k \varphi_{r-k}(n)}{k!} = \frac{r^n}{(r-1)!}.$$

Proof. From (6.1) and (3.3),

$$e^x \sum_{k=1}^{n+1} \varphi_k(n) x^k = \sum_{k=1}^{\infty} \frac{k^n x^k}{(k-1)!}. \tag{6.2}$$

Equating coefficients of x^r on both sides above, we achieve the sought equality.

Entry 7. ***Let r and n be nonnegative integers with*** $r \leq n$. ***Then***

$$r!\, \varphi_{r+1}(n) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} (r+1-k)^n.$$

Proof. Multiply both sides of (6.2) by e^{-x} and equate coefficients of x^{r+1} on both sides to reach the desired conclusion.

Entry 8. ***Let n and r be integers such that*** $1 \leq r \leq n+1$. ***Then***

$$\varphi_r(n+1) = r\varphi_r(n) + \varphi_{r-1}(n),$$

where $\varphi_0(n) = 0$.

Proof. By Entry 7,

$$\begin{aligned}
\varphi_r(n+1) - \varphi_{r-1}(n) &= \frac{1}{(r-1)!} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} (r-k)^{n+1} \\
&\quad - \frac{1}{(r-2)!} \sum_{k=1}^{r-1} (-1)^{k-1} \binom{r-2}{k-1} (r-k)^n \\
&= \frac{1}{(r-1)!} \left\{ r^{n+1} + \sum_{k=1}^{r-1} (-1)^k \left\{ (r-k)\binom{r-1}{k} \right.\right. \\
&\quad \left.\left. + (r-1)\binom{r-2}{k-1} \right\} (r-k)^n \right\} \\
&= \frac{1}{(r-1)!} \left\{ r^{n+1} + r \sum_{k=1}^{r-1} (-1)^k \binom{r-1}{k} (r-k)^n \right\} \\
&= r\varphi_r(n).
\end{aligned}$$

Ramanujan next indicates that the recursion formula in Entry 8 can be employed to calculate $f_n(x)$. In the following corollary, Ramanujan gives $f_n(x)$, $0 \leq n \leq 6$. In a corollary after Entry 5, Ramanujan inexplicitly indicates that the calculus of finite differences in conjunction with Entry 5 can also be used to calculate $f_n(x)$. Since this is now very well known, we shall forego any further calculations and be content with merely exhibiting the first seven polynomials.

Corollary.

$$
\begin{aligned}
f_0(x) &= x,\\
f_1(x) &= x + x^2,\\
f_2(x) &= x + 3x^2 + x^3,\\
f_3(x) &= x + 7x^2 + 6x^3 + x^4,\\
f_4(x) &= x + 15x^2 + 25x^3 + 10x^4 + x^5,\\
f_5(x) &= x + 31x^2 + 90x^3 + 65x^4 + 15x^5 + x^6,\\
f_6(x) &= x + 63x^2 + 301x^3 + 350x^4 + 140x^5 + 21x^6 + x^7.
\end{aligned}
$$

Example 1. *Let $\varphi_1(n), \ldots, \varphi_{n+1}(n)$, $0 \le n < \infty$, be defined by* (6.1). *Let $\{a_k\}$, $1 \le k < \infty$, be any sequence of complex numbers such that*

$$\psi(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a_k}{(z+1)(z+2)\cdots(z+k)}$$

has abscissa of convergence $\lambda < \infty$. Define, for each nonnegative integer j, $F(j) = \sum_{k=1}^{j+1} a_k \varphi_k(j)$. Let R_ε be defined by (3.2) if $\lambda = -\infty$, but if λ is finite, let

$$R_\varepsilon = \{z\colon -\tfrac{1}{2}\pi + \varepsilon \le \arg(z-\lambda) \le \tfrac{1}{2}\pi - \varepsilon\}, \qquad \varepsilon > 0.$$

Then as z tends to ∞ in R_ε,

$$\psi(z) \sim \sum_{k=0}^{\infty} \frac{(-1)^k F(k)}{z^{k+1}}.$$

Proof. Using a well-known generating function for Stirling numbers of the second kind found in the handbook of Abramowitz and Stegun [1, formula 24.1.4B, p. 824], we have

$$
\begin{aligned}
\frac{1}{(z+1)(z+2)\cdots(z+k)} &= \sum_{j=k}^{\infty} \frac{(-1)^{j+k}\varphi_k(j-1)}{z^j}\\
&= \sum_{j=k}^{n} \frac{(-1)^{j+k}\varphi_k(j-1)}{z^j} + O(z^{-n-1})
\end{aligned}
$$

as z tends to ∞ in R_ε. Hence,

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{(-1)^{k-1} a_k}{(z+1)(z+2)\cdots(z+k)} &= \sum_{k=1}^{n} (-1)^{k-1} a_k \sum_{j=k}^{n} \frac{(-1)^{j+k}\varphi_k(j-1)}{z^j} + O(z^{-n-1})\\
&= \sum_{j=1}^{n} \frac{(-1)^{j-1}}{z^j} \sum_{k=1}^{j} a_k \varphi_k(j-1) + O(z^{-n-1})\\
&= \sum_{j=1}^{n} \frac{(-1)^{j-1} F(j-1)}{z^j} + O(z^{-n-1}). \qquad (8.1)
\end{aligned}
$$

This completes the proof.

Example 2. Let r and n be integers with $0 \le r \le n$. Then $r!\,\varphi_{r+1}(n)$ is the *coefficient of $x^n/n!$ in the Maclaurin series of $e^x(e^x - 1)^r$.*

Proof. By Entry 7, $r!\,\varphi_{r+1}(n)$ is the coefficient of $x^n/n!$ in the expansion of

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k} e^{x(r+1-k)} = e^x(e^x - 1)^r,$$

and the proof is complete.

Example 3. *Let n be a positive integer. Then*

$$f'_{n-1}(x) = \sum_{k=0}^{n-1} \binom{n}{k} f_{k-1}(x).$$

Proof. From Entry 4,

$$\sum_{n=0}^{\infty} \frac{a^{n+1}}{(n+1)!} f'_n(x) = (e^a - 1)e^{x(e^a-1)} = (e^a - 1) \sum_{k=0}^{\infty} \frac{a^k}{k!} f_{k-1}(x).$$

Equating the coefficients of a^n on both sides, we complete the proof.

Both Examples 2 and 3 are well known.

The next example is the first of many entries in the second notebook that involves the Bernoulli numbers B_n, $0 \le n < \infty$. Ramanujan defines the Bernoulli numbers by

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} x^{2n}, \qquad |x| < 2\pi. \tag{8.2}$$

However, today the Bernoulli numbers are more commonly defined by (Il), and so the latter convention shall be employed here. Moreover, generally, Ramanujan's formulas are more easily stated in the notation (Il) rather than (8.2).

Example 4. *Let n denote an integer greater than or equal to* -1. *Then*

$$\int_0^x f_n(t)\,dt = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{B_{n+1-k} f_k(x)}{k+1}.$$

Proof. Replace x by t in (4.1) and integrate both sides over $0 \le t \le x$ to obtain, for *(a(< 271,*

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} \int_0^x f_{n-1}(t)\,dt = \frac{1}{e^a - 1}\{e^{x(e^a-1)} - 1\}$$

$$= \frac{1}{a} \sum_{j=0}^{\infty} \frac{B_j a^j}{j!} \sum_{k=1}^{\infty} \frac{a^k}{k!} f_{k-1}(x)$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{k=1}^{n+1} \binom{n}{k-1} \frac{B_{n+1-k} f_{k-1}(x)}{k} \right\} \frac{a^n}{n!}, \tag{8.3}$$

where we have utilized (Il). Now equate the coefficients of a^{n+1} on the extremal sides of (8.3) to obtain the desired result.

Example 4 was incorrectly stated by Ramanujan. On the right side of his equality, replace n by $n + 1$ everywhere except in the suffixes.

Example 5(i). *For each* nonnegative integer n, dejine *A, by*

$$eA_n = ef_n(1) = \sum_{k=1}^{\infty} \frac{k^n}{(k-1)!}. \tag{8.4}$$

Then *A,* = 1, *A,* = 2, *A,* = *5, A,* = *15, A,* = *52, A,* = 203, *A,* = *877, A ,* = 4140, and *A ,* = 21147.

Proof. It is not difficult to show that, with x = 1, (5.1) can be equivalently expressed by means of difference notation in the form

$$\Delta^n A_n = A,-_1, \qquad n \geq 1. \tag{8.5}$$

Since *A,* = 1, (8.5) can be used to construct a difference table in order to calculate *A,*

The numbers *A,* are now called Bell numbers with the nth Bell number B(n) being defined by B(n) = A_{n-1}, $n \geq 1$. Combinatorially, *B(n)* is the number of ways of partitioning a set of n elements. According to Gould [9], the earliest known application of these numbers is in an edition of the Japanese Tale *of Genji* published in the seventeenth Century. The numbers B(n) arose as the number of ways of arranging n incense sticks. As another application, Browne [1] observed that *B(n)* is the number of ways of rhyming a stanza of n lines. The first explicit appearance of these numbers apparently is in a paper of C. Kramp [1] in 1796. They are also found in a treatise of J. Tate [1, p. 45] published in 1845. The formula (8.4) appears as a problem [1] in the Matematicheskii Sbornik in 1868. In 1877, Dobiński [1] used (8.4) to calculate B(l), . ..) *B(8)*. In 1885, Cesàro [1] found the numbers to be solutions of the difference equation (8.5). Again, in connection with (8.4), the numbers appear in problems in the texts of Hardy [16, p. 424] and Bromwich [1, p. 197]. Touchard [1], [2], Bell [1], Browne [1], Williams [1], Ginsburg [1], and Balasubrahmanian [1] have established several elementary properties and give lists of varying lengths of the Bell numbers. Carlitz [8] has written a nice paper on Bell numbers, Stirling numbers of the second kind, and some generalizations. For references to other papers of Carlitz on this subject see his paper [8]. Levine and Dalton [1] have calculated the first 74 Bell numbers. Obviously, *B(n)* grows very rapidly, and Epstein [1] has found an asymptotic formula for *B(n).* He has also discovered other analytic properties of the Bell numbers, for example, integral representations. For the numbers listed in Example 5(i), *A,* is even if $n \equiv 1 \pmod 3$, and *A,,* is odd otherwise. This property persists, and a simple proof of it can be found in the paper of

Balasubrahmanian [1]. Actually, more general congruences are known; see papers of Touchard [2] and Williams [1], for example. The Bell numbers have been rediscovered by many authors, and we have listed but a small portion of those papers in which properties of the Bell numbers are proved and combinatorial applications are given. For further references, readers should consult Gould's extremely comprehensive bibliography [9].

Example 5(ii). *For each nonnegative integer n, define C_n by*

$$\frac{C_n}{e} = \sum_{k=1}^{\infty} \frac{(-1)^k k^n}{(k-1)!}.$$

Then $C_0 = -1$, $C_1 = 0$, $C_2 = C_3 = 1$, $C_4 = -2$, $C_5 = C_6 = -9$, $C_7 = 50$, *and* $C_8 = 267$.

Proof. Observe that $C_n = f_n(-1)$, $n \geq 0$. This, from (5.1) it is readily shown that $\Delta^n C_n = -C_{n-1}$, $n \geq 1$. Using this difference equation and the initial value $C_0 = -1$, we may compose a difference table to calculate C_n.

The equalities in the next example are easily verified from Examples 5(i) and 5(ii).

Example 6.

(i) $$f_3(1) = 3f_2(1) = 15,$$

(ii) $$f_5(1) + f_2(1) = 4f_4(1) = 208,$$

(iii) $$f_3(-1) = f_2(-1) = 1,$$

(iv) $$f_6(-1) = f_5(-1) = -9,$$

(v) $$f_8(-1) + f_6(-1) + f_5(-1) + f_3(-1) = 5f_7(-1) = 250.$$

Let x, a, and b be complex numbers, and let n be a nonnegative integer. Generalizing $f_n(x)$, we define

$$e^x F_n(a, b; x) = e^x F_n(x) = \sum_{k=1}^{\infty} \frac{(a + bk)^n x^k}{(k-1)!}. \tag{9.1}$$

Thus, $f_n(x)$, defined by (3.3), is the particular case of $F_n(x)$ which is obtained by putting a = 0 and b = 1. Moreover, $F_n(x)$ can readily be expressed in terms of $f_0(x), \ldots, f_n(x)$, since

$$e^x F_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j k^j$$

$$= e^x \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j f_j(x). \tag{9.2}$$

Expressed in a slightly different way, Entry 9(i) below is a generalization of Entry 3.

Entry 9(i). ***As z tends to ∞ in R_ε, where R_ε is defined by*** (3.2),

$$\sum_{k=0}^{n} \frac{(-1)^k F_k(x)}{z^{k+1}} = \sum_{k=1}^{n+1} \frac{(-b)^{k-1}x^k}{(z+a+b)(z+a+2b)\cdots(z+a+kb)} + O(z^{-n-2}).$$

Proof. By Taylor? theorem,

$$\sum_{k=j}^{n} \binom{k}{j} \frac{(-1)^{k+j}a^k}{z^{k+1}} = \frac{a^j}{z^{j+1}}\left(1+\frac{a}{z}\right)^{-j-1} + O(z^{-n-2}), \tag{9.3}$$

as z tends to ∞. Thus, by (9.2), (9.3), and (8.1),

$$\begin{aligned}\sum_{k=0}^{n} \frac{(-1)^k F_k(x)}{z^{k+1}} &= \sum_{k=0}^{n} \frac{(-1)^k}{z^{k+1}} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} b^j f_j(x)\\ &= \sum_{j=0}^{n} \left(\frac{-b}{a}\right)^j f_j(x) \sum_{k=j}^{n} \binom{k}{j} \frac{(-1)^{j+k}a^k}{z^{k+1}}\\ &= \sum_{j=0}^{n} \frac{(-b)^j}{(z+a)^{j+1}} f_j(x) + O(z^{-n-2})\\ &= \frac{1}{b}\sum_{k=1}^{n+1} \frac{(-1)^{k-1}x^k}{\left(\frac{z+a}{b}+1\right)\left(\frac{z+a}{b}+2\right)\cdots\left(\frac{z+a}{b}+k\right)} + O(z^{-n-2}),\end{aligned}$$

from which the desired asymptotic formula follows.

The next entry generalizes Entry 4.

Entry 9(ii). ***If a, b, x, and y are complex numbers, we have***

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} F_n(x) = xe^{(a+b)y}e^{x(e^{by}-1)}. \tag{9.4}$$

Proof. By (9.2),

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{y^n}{n!} F_n(x) &= \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k f_k(x)\\ &= \sum_{k=0}^{\infty} \left(\frac{b}{a}\right)^k f_k(x) \sum_{n=k}^{\infty} \binom{n}{k} \frac{(ay)^n}{n!}\\ &= \sum_{k=0}^{\infty} \frac{(by)^k e^{ay}}{k!} f_k(x).\end{aligned}$$

Applying (5.2), with by in place of a, we complete the proof.

Entry 9(iii). *For each nonnegative integer n, we have*

$$F_{n+1}(x) - (a+b)F_n(x) = bx \sum_{k=0}^{n} \binom{n}{k} b^k F_{n-k}(x).$$

Proof. Differentiating both sides of (9.4) with respect to y, we find that

$$\sum_{n=1}^{\infty} \frac{y^{n-1}}{(n-1)!} F_n(x) = (a + b + bxe^{by}) \sum_{n=0}^{\infty} \frac{y^n}{n!} F_n(x).$$

On equating coefficients of y” on both sides, we finish the proof.

Entry 9(iii) is obviously an analogue of Entry 5. After Entry 9(iv), Ramanujan indicates very briefly how to express Entry 9(iii) in terms of differences. For each nonnegative integer *n,* define

$$\psi_n(x) = F_{n+1}(x) \quad (a+b)F_n(x) = bx(b+F)^n,$$

where in the expansion of *(b* + *F)“*, F^k is to be interpreted as meaning $F_k(x)$. Next, define an operator δ by $\delta g(n) = g(n) - bg(n-1)$. *So,*

$$\delta\psi_n = \psi_n - b\psi_{n-1} = bx(b+F)^{n-1}F.$$

By inducting on *k,* it can easily be shown that

$$\delta^k\psi_n = bx(b+F)^{n-k}F^k, \qquad 0 \le k \le n.$$

In particular,

$$\delta^n\psi_n = bxF^n = bxF_n(x).$$

Since *F,(x)* = x, it follows from Entry 9(iii), or from (9.2), or from the preceding paragraph, that *F,(x)* is a polynomial in x of degree *n* + 1. Moreover, *F,(O) = 0.* Hence, we define $\varphi_1(n), \ldots, \varphi_{n+1}(n)$, $n \ge 0$, by

$$F_n(x) = \sum_{k=1}^{n+1} \varphi_k(n)x^k. \tag{9.5}$$

The next four results generalize Entries 6, 7, and 8, and Example 2 of Section 8, respectively. The proofs are completely analogous, and so we omit them.

Entry 9(iv). *Suppose that r and n are integers such that* $0 < r \le n + 1$. *Then*

$$\sum_{k=0}^{r-1} \frac{\varphi_{r-k}(n)}{k!} = \frac{(a+br)^n}{(r-1)!}.$$

Entry 9(v). *Let r and n be as in Entry* 9(iv). *Then*

$$(r-1)!\,\varphi_r(n) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \{a + (r-k)b\}^n.$$

Entry 9(vi). ***Let n and r be integers such that*** $1 \le r \le n+1$. ***Then***

$$\varphi_r(n+1) = (a+br)\varphi_r(n) + b\varphi_{r-1}(n),$$

where $\varphi_0(n) = 0$.

Using Entry 9(vi), Ramanujan next calculates $F_n(x)$, $1 \le n \le 4$. Thus,

$$F_0(x) = x,$$

$$F_1(x) = (a+b)x + bx^2,$$

$$F_2(x) = (a+b)^2x + b(2a+3b)x^2 + b^2x^3,$$

$$F_3(x) = (a+b)^3x + b\{3(a+b)(a+2b) + b^2\}x^2 + 3b^2(a+2b)x^3 + b^3x^4,$$

$$F_4(x) = (a+b)^4x + b(2a+3b)\{2(a+b)(a+26) + b^2\}x^2 + b^2\{6(a+2b)^2 + b^2\}x^3 + 2b^3(2a+5b)x^4 + b^4x^5. \quad (9.6)$$

Entry 9(vii). ***Let r and n be integers with*** $0 \le r \le n$. ***Then*** $r!\,\varphi_{r+1}(n)$ ***is the coefficient of*** $x^n/n!$ ***in the Maclaurin series of*** $e^{(a+b)x}(e^{bx}-1)^r$.

Example (i).

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^3 + (2k+1)^2}{k!} = 0$$

Proof. From (9.1), we have

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^3 + (2k+1)^2}{k!} = -\frac{1}{e}\{F_3(-1, 2; -1) + F_2(-1, 2; -1)\}. \quad (9.7)$$

But from (9.6), $F_2(-1, 2; x) = x + 8x^2 + 4x^3$ and $F_3(\ 1, 2; x) = x + 26x^2 + 36x^3 + 8x^4$. Hence, $F_2(-1, 2; -1) = 3 = -F_3(\ 1, 2; -1)$. Using these values in (9.7), we complete the proof.

Example (ii).

$$\sum_{k=1}^{\infty} \frac{k^4}{(k-1)!} = 4\sum_{k=0}^{\infty} \frac{(2k+1)^2}{k!}.$$

Proof. The left side above is 52e by Example 5(i) of Entry 8. The right side above is $4eF_2(-1, 2; 1)$ by (9.1). But from the previous proof, $F_2(\ 1, 2; 1) = 13$, and so the proof is complete.

Example (iii).

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k^7 + k^6}{(k-1)!} = \sum_{k=1}^{\infty} (\ \ 1)^k \frac{(2k-1)^4}{(k-1)!}.$$

Proof. By Example 5(ii) of Entry 8, the left side above is $-41/e$. Now, by (9.6), $F_4(-1, 2; x) = x + 80x^2 + 232x^3 + 128x^4 + 16x^5$. Thus, the right side above is $F_4(-1, 2; -1)/e = -41/e$.

Example (iii) must be corrected in the second notebook by multiplying either side of the equality by -1. Example (iv) must be corrected in the second notebook by replacing -4 on the right side of the equality by -8.

Example (iv). $$\sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)^4}{k!\,k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k} - 8.$$

Proof. We have

$$\sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)^4}{k!\,k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k} + \sum_{k=1}^{\infty} (-1)^k \frac{8 + 24k + 32k^2 + 16k^3}{k!}$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k} - \frac{1}{e}\{8F_0(-1, 1; -1) + 24F_1(-1, 1; -1)$$
$$+ 32F_2(-1, 1; -1) + 16F_3(-1, 1; -1)\} - 8. \qquad (9.8)$$

Now, from (9.6), $F_0(x) = x$, $F_1(x) = x^2$, $F_2(x) = x^2 + x^3$, and $F_3(x) = x^2 + 3x^3 + x^4$ when $-a = 1 = b$. Thus, the expression in curly brackets on the right side of (9.8) is equal to 0. This completes the proof.

Some properties of $F_n(a-1, 1; x)$ have been derived by Manikarnikamma [1].

In preparation for Entry 10, we first define a sequence of nonnegative integers b_{kn}, $k \geq 2$, by the equalities:

$$\begin{cases} b_{kk} = 1;\ b_{kn} = 0, & \text{for } n < k \text{ or } n > 2k-2; \text{ and} \\ b_{k+1,n+1} = nb_{k,n-1} + (n-k+1)b_{kn}, & \text{for } k \leq n \leq 2k-1. \end{cases} \qquad (10.1)$$

A short table of values for b_{kn} is provided below.

k \ n	2	3	4	5	6	7	8	9	10	11	12
2	1										
3		1	3								
4			1	10	15						
5				1	25	105	105				
6					1	56	490	1260	945		
7						1	119	1918	9450	17325	10395

In fact, $b_{kn} = S_2(n, n+1-k)$, where $S_2(n, k)$ is the 2-associated Stirling number of the second kind. (See the books of Comtet [1, pp. 221–222] and Riordan [1, pp. 74–78].)

Entry 10. *Let $\varphi(x)$ denote a function of at most polynomial growth as x (real) tends to ∞. Suppose that there exists a constant $A \geq 1$ and a function $G(x)$ of at most polynomial growth as x tends to ∞ such that for each nonnegative integer* *m and all sufficiently large x, the derivatives* $\varphi^{(m)}(x)$ *exist and satisfy*

$$\left|\frac{\varphi^{(m)}(x)}{m!}\right| \leq G(x)\left(\frac{A}{x}\right)^m. \tag{10.2}$$

Put

$$\varphi_\infty(x) = e^{-x} \sum_{k=0}^{\infty}{}' \frac{x^k\varphi(k)}{k!},$$

where the prime on the summation sign indicates that the (finitely many) terms for which $\varphi(k)$ *may be undejned are not included in the sum. Then for any fixed positive integer M,*

$$q,(x) = \varphi(x) + \sum_{k=2}^{M} \sum_{n=k}^{2k-2} b_{kn}x^{n-k+1} \frac{\varphi^{(n)}(x)}{n!} + O(G(x)x^{-M}), \tag{10.3}$$

as x tends to CO, where the numbers b_{kn} *are defined by* (10.1).

Before embarking upon a proof of Entry 10, we offer several comments. Examples of functions φ satisfying the conditions of Entry 10 are functions of polynomial growth that are analytic in some right half plane. This follows from Cauchy's integral formula for derivatives. Specific examples will be given upon the conclusion of the proof of Entry 10.

The following result is related to Entry 10. If φ is bounded and continuous on $[0, \infty]$, then from Feller's text [1, pp. 219, 227]

$$e^{-x} \sum_{k=0}^{\infty} \frac{x^k\varphi(k)}{k!} \sim \varphi(x),$$

as x tends to ∞. Observe that the left side above is the expected value $E(\varphi(U))$, where U is a random variable with Poisson distribution of mean x.

The asymptotic formula above has a superficial resemblance to Bore1 summability. However, it is doubtful that Ramanujan was influenced by this. In particular, no other material in the second notebook pertains to Bore1 summability.

Formula (10.3) is a more precise version of the formula that Ramanujan gives in his Entry 10. He provides a very brief sketch of his forma1 "proof" of Entry 10, amd because it is instructive, we shall give it below.

Ramanujan tacitly assumes that φ is an entire function. Hence,

$$\begin{aligned} e^x\varphi_\infty(x) &= \sum_{k=0}^{\infty} \frac{x^k\varphi(k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)k^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{\infty} \frac{k^{n-1}x^k}{(k-1)!} \\ &= e^x \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)f_{n-1}(x)}{n!}, \end{aligned}$$

by (3.3), where it is assumed that the inversion in order of summation above is justified. (There is a misprint in the notebooks in that $f_{n-1}(x)$ is replaced by $f_n(x)$.) Using (6.1) above, we find that

$$\varphi_\infty(x) = \varphi(0) + \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{j=1}^{n} \varphi_j(n-1)x^j$$

$$= \varphi(0) + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-k+1}(n-1)x^{n-k+1}. \tag{10.4}$$

We now separate $\varphi(0)$ together with the series for $k = 1$ in (10.4). These terms are

$$\varphi(0) + \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_n(n-1)x^n = \varphi(0) + \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{n!} x^n = \varphi(x). \tag{10.5}$$

Here we have used the fact that

$$\varphi_n(n-1) = 1, \qquad n \geq 1, \tag{10.6}$$

which is easily proved by induction with the aid of Entry 8.

Next, we examine the series for $k = 2$ in (10.4). This series is

$$\sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-1}(n-1)x^{n-1} = \frac{x}{2} \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{(n-2)!} x^{n-2} = \frac{x}{2} \varphi''(x). \tag{10.7}$$

In this calculation we have used the evaluation $\varphi_n(n) = n(n+1)/2$ for $n \geq 1$, which again is readily established by induction with the help of Entry 8.

Ramanujan continues to calculate in the fashion indicated above. In fact, using special cases of Lemma 3 below, he finds that

$$\sum_{n=3}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-2}(n-1)x^{n-2} = \frac{x}{6} \varphi'''(x) + \frac{x^2}{8} \varphi^{(4)}(x), \tag{10.8}$$

$$\sum_{n=4}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-3}(n-1)x^{n-3} = \frac{x}{24} \varphi^{(4)}(x) + \frac{x^2}{12} \varphi^{(5)}(x) + \frac{x^3}{48} \varphi^{(6)}(x), \tag{10.9}$$

$$\sum_{n=5}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-4}(n-1)x^{n-4}$$

$$= \frac{x}{120} \varphi^{(5)}(x) + \frac{5x^2}{144} \varphi^{(6)}(x) + \frac{x^3}{48} \varphi^{(7)}(x) + \frac{x^4}{384} \varphi^{(8)}(x), \tag{10.10}$$

and

$$\sum_{n=6}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-5}(n-1)x^{n-5}$$

$$= \frac{x}{720}\varphi^{(6)}(x) + \frac{x^2}{90}\varphi^{(7)}(x) + \frac{7x^3}{576}\varphi^{(8)}(x) + \frac{x^4}{288}\varphi^{(9)}(x) + \frac{x^5}{3840}\varphi^{(10)}(x). \tag{10.11}$$

At this point Ramanujan ceases his calculations and substitutes (10.5) and (10.7)–(10.11) into (10.4). With the help of the table for b_{kn}, we readily verify that Ramanujan's result agrees with (10.3).

In a corollary, Ramanujan claims that $\varphi_\infty(x) = \varphi(x) + (x/2)\varphi''(x)$ "very nearly." However, no discussion of the error term is given.

Before commencing the proof of Entry 10, we provide four lemmas.

Lemma 1. ***Let t be fixed, where $0 < t < 1$. Then***

$$e^{-x} \sum_{0 \le k < tx} \frac{x^k}{k!} \quad \text{and} \quad e^{-x} \sum_{k > x/t} \frac{x^k}{k!}$$

each tend to 0 exponentially as x tends to ∞.

An easy proof of Lemma 1 has been given by Breusch in a solution to a problem posed by Moy [1].

Lemma 2. ***Let $2 \le k \le n$. Then $b_{kn} \le (n-1)!$.***

Proof. With the use of (10.1), induct on n, and the result follows easily.

Lemma 3. ***Let $2 \le k \le n$. Then***

$$\varphi_{n+1-k}(n-1) = \sum_{j=k}^{2k-2} b_{kj} \binom{n}{j},$$

where $\varphi_k(n)$ is defined by (6.1).

Proof. The result follows from Entry 8 by induction on k. See also Comtet's text [1, p. 226].

Lemma 4. ***Let p(x) be a polynomial of degree n. Then***

$$p_\infty(x) \equiv e^{-x} \sum_{k=0}^{\infty} \frac{x^k p(k)}{k!} = p(x) + \sum_{j=2}^{n} \sum_{k=2}^{j} b_{kj} x^{j-k+1} \frac{p^{(j)}(x)}{j!}.$$

Proof. By linearity, it suffices to prove this lemma in the case that $p(x) = x^n$, where n is a nonnegative integer. The result is easily proved for $n = 0$, 1, and so we suppose that $n \ge 2$. By (6.1),

$$p_\infty(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k k^n}{k!} = f_{n-1}(x) = \sum_{k=1}^{n} \varphi_k(n-1) x^k$$

$$= \varphi_n(n-1)x^n + \sum_{k=2}^{n} \varphi_{n-k+1}(n-1) x^{n-k+1}.$$

Using (10.6) and Lemma 3, we deduce that

$$p_\infty(x) = x^n + \sum_{k=2}^{n} \sum_{j=k}^{2k-2} b_{kj} x^{j-k+1} \binom{n}{j} x^{n-j}$$

$$= p(x) + \sum_{k=2}^{n} \sum_{j=k}^{2k-2} b_{kj} x^{j-k+1} \frac{p^{(j)}(x)}{j!}.$$

Since $p^{(j)}(x) = 0$ for $j > n$ and since $b_{kj} = 0$ for $j > 2k - 2$, the upper index $2k - 2$ on the inner sum may be replaced by n. The result now follows upon inverting the order of summation.

Proof of Entry 10. Throughout the proof we always assume that x is sufficiently large. Fix $t \in (0, 1)$, but we require that t is close enough to 1 so that $3(1 - t)A/t < 1$. Define the intervals I_1, I_2, and I_3 by $I_1 = [0, tx)$, $I_2 = [tx, (2 - t)x)$, and $I_3 = [(2 - t)x, \infty)$.

Consider the Taylor polynomial

$$p(y) = \sum_{r=0}^{N-1} \frac{\varphi^{(r)}(x)}{r!}(y - x)^r, \tag{10.12}$$

where $N = [\sqrt{x}/6A]$. By Taylor's theorem, for each $y \in I_2$,

$$\varphi(y) = p(y) + \frac{\varphi^{(N)}(\xi)}{N!}(y - x)^N,$$

where ξ is some point between x and y. Thus, by (10.2),

$$\begin{aligned}|\varphi(y) - p(y)| &\le \left|\frac{\varphi^{(N)}(\xi)}{N!}\right| \{x(1 - t)\}^N \\ &\le G(\xi)\left(\frac{A}{tx}\right)^N \{x(1 - t)\}^N \\ &< G(\xi)3^{-N} < 2^{-N},\end{aligned}$$

for every $y \in I_2$. Therefore, as x tends to ∞,

$$e^{-x} \sum_{k \in I_2} \frac{x^k \varphi(k)}{k!} = e^{-x} \sum_{k \in I_2} \frac{x^k p(k)}{k!} + O(2^{-N}). \tag{10.13}$$

Since $\varphi(x)$ has at most polynomial growth as x tends to ∞, it follows from Lemma 1 that

$$e^{-x} \sum_{k \in I_1}' \frac{x^k \varphi(k)}{k!} \to 0 \quad \text{exponentially as } x \to \infty. \tag{10.14}$$

Also, for some fixed natural number B,

$$e^{-x} \sum_{k \in I_3} \frac{x^k \varphi(k)}{k!} \ll e^{-x} \sum_{k \in I_3} \frac{x^k}{(k - B)!} = x^B e^{-x} \sum_{k \ge (2-t)x - B} \frac{x^k}{k!}.$$

So again by Lemma 1,

$$e^{-x} \sum_{k \in I_3} \frac{x^k \varphi(k)}{k!} \to 0 \quad \text{exponentially as } x \to \infty. \tag{10.15}$$

By (10.2) and (10.12) for $0 \le y \le 2x$,

$$|p(y)| \le \sum_{r=0}^{N-1} G(x)\left(\frac{A}{x}\right)^r x^r \ll (A+1)^N, \tag{10.16}$$

and so by Lemma 1,

$$e^{-x}\sum_{k\in I_1}\frac{x^k p(k)}{k!} \to 0 \quad \text{exponentially as } x\to\infty. \tag{10.17}$$

Write

$$e^{-x}\sum_{k\in I_3}\frac{x^k p(k)}{k!} = S_1 + S_2, \tag{10.18}$$

where

$$S_1 = e^{-x}\sum_{(2-t)x\le k<2x}\frac{x^k p(k)}{k!} \quad \text{and} \quad S_2 = \sum_{k\ge 2x}\frac{x^k p(k)}{k!}.$$

By (10.16) and Lemma 1,

$$S_1 \to 0 \quad \text{exponentially as } x\to\infty. \tag{10.19}$$

For $k \ge 2x$, it follows from (10.2) that

$$\begin{aligned} |p(k)| &\le \sum_{r=0}^{N-1} G(x)\left(\frac{A}{x}\right)^r (k-x)^r \\ &\le G(x)NA^N\left(\frac{k-x}{x}\right)^N \ll \left(\frac{k}{x}\right)^{x/7}. \end{aligned}$$

Also, for $k \ge 2x$, $x^k/k! < (xe/k)^k$. Hence,

$$S_2 \ll e^{-x}\sum_{k\ge 2x}\left(\frac{xe}{k}\right)^k\left(\frac{k}{x}\right)^{x/7}. \tag{10.20}$$

The summands in (10.20) are strictly decreasing in k. Thus,

$$\begin{aligned} S_2 &\ll e^{-x}\sum_{j=2}^{\infty}\sum_{jx\le k<(j+1)x}\left(\frac{xe}{k}\right)^k\left(\frac{k}{x}\right)^{x/7} \\ &\ll xe^{-x}\sum_{j=2}^{\infty}\left(\frac{xe}{jx}\right)^{jx}\left(\frac{jx}{x}\right)^{x/7} \\ &= x\sum_{j=2}^{\infty}\{e^{j-1}j^{-j+1/7}\}^x \\ &< x\left(\sum_{j=2}^{\infty}\{e^{j-1}j^{-j+1/7}\}^3\right)^{x/3}. \end{aligned}$$

Since the series in parentheses above converges to a number less than 1,

$$S_2 \to 0 \quad \text{exponentially as } x\to\infty. \tag{10.21}$$

By (10.18), (10.19), and (10.21),

$$e^{-x}\sum_{k\in I_3}\frac{x^k p(k)}{k!}\to 0 \quad \text{exponentially as } x\to\infty. \tag{10.22}$$

By (10.13), (10.14), (10.15), (10.17), and (10.22),

$$\varphi_\infty(x)-p_\infty(x)=O(2^{-N}) \tag{10.23}$$

as x tends to ∞, where

$$p_\infty(x)=e^{-x}\sum_{k=0}^{\infty}\frac{x^k p(k)}{k!}.$$

By Lemma 4, (10.12), and (10.1),

$$\begin{aligned}
p_\infty(x) &= p(x)+\sum_{j=2}^{N-1}\sum_{k=2}^{j} b_{kj}x^{j-k+1}\frac{p^{(j)}(x)}{j!}\\
&=\varphi(x)+\sum_{j=2}^{N-1}\sum_{k=2}^{j} b_{kj}x^{j-k+1}\frac{\varphi^{(j)}(x)}{j!}\\
&=\varphi(x)+\sum_{k=2}^{N-1}\sum_{j=k}^{N-1} b_{kj}x^{j-k+1}\frac{\varphi^{(j)}(x)}{j!}\\
&=\varphi(x)+\sum_{k=2}^{M}\sum_{j=k}^{2k-2} b_{kj}x^{j-k+1}\frac{\varphi^{(j)}(x)}{j!}+S_3,
\end{aligned} \tag{10.24}$$

where

$$S_3=\sum_{k=M+1}^{N-1}\sum_{j=k}^{N-1} b_{kj}x^{j-k+1}\frac{\varphi^{(j)}(x)}{j!}.$$

In view of (10.23) and (10.24), in order to prove (10.3) it suffices to show that $S_3=O(G(x)x^{-M})$ as x tends to ∞.

By (10.1), (10.2), and Lemma 2,

$$\begin{aligned}
|S_3| &\le \sum_{k=M+1}^{N}\sum_{j=k}^{2k-2} b_{kj}x^{j-k+1}\frac{|\varphi^{(j)}(x)|}{j!}\\
&\le \sum_{k=M+1}^{N}\sum_{j=k}^{2k-2}(j-1)!\,G(x)A^j x^{1-k}\\
&\le G(x)\sum_{k=M+1}^{N}(2k)!\,A^{2k}x^{1-k}\\
&\le G(x)x^{-M}\sum_{k=0}^{N}(2k+2m+2)!\left(\frac{A}{\sqrt{x}}\right)^{2k}\\
&\le G(x)x^{-M}\sum_{k=0}^{2N}(k+2M+2)!\left(\frac{A}{\sqrt{x}}\right)^{2k}.
\end{aligned}$$

Since $N \leq \sqrt{x}/6A$, the $(k+1)$th term in the last sum above is less than half of the kth term, for each $k < 2N$. Thus,

$$|S_3| \leq G(x)x^{-M}(2M+2)! \sum_{k=0}^{2N} 2^{-k} = O(G(x)x^{-M}).$$

This completes the proof.

The following four examples give applications of (10.3).

Example 1. *As x tends to* ∞,

$$\operatorname{Log}\left(\sum_{k=1}^{\infty} \frac{x^k\sqrt{k}}{k!}\right) = x + \tfrac{1}{2}\operatorname{Log} x - \frac{1}{8x} - \frac{1}{16x^2} + O\left(\frac{1}{x^3}\right).$$

Proof. Letting $\varphi(x) = G(x) = \sqrt{x}$ in (10.3) with $M = 3$, we find that

$$e^{-x}\sum_{k=1}^{\infty} \frac{x^k\sqrt{k}}{k!} = \sqrt{x}\left(1 - \frac{1}{8x} - \frac{7}{128x^2} + O\left(\frac{1}{x^3}\right)\right),$$

as x tends to ∞. Hence,

$$\operatorname{Log}\left(\sum_{k=1}^{\infty} \frac{x^k\sqrt{k}}{k!}\right) = \operatorname{Log}\left\{e^x\sqrt{x}\left(1 - \frac{1}{8x} - \frac{7}{128x^2} + O\left(\frac{1}{x^3}\right)\right)\right\},$$

and the result easily follows.

Example 1 is actually a special case of a result of Hardy [4, pp. 410–411], [18, pp. 71–72], who derived in the case $\varphi(x) = (x+a)^{-s}$ an asymptotic series which is in a more complicated form than that given by (10.3). In particular, we find that

$$\begin{aligned} e^{-x}\sum_{k=0}^{\infty}{}' \frac{x^k(k+a)^{-s}}{k!} = (x+a)^{-s} &+ \frac{s(s+1)}{2}x(x+a)^{-s-2} \\ &- \frac{s(s+1)(s+2)}{6}x(x+a)^{-s-3} \\ &+ \frac{s(s+1)(s+2)(s+3)}{8}x^2(x+a)^{-s-4} \\ &+ O(x^{-s-3}). \end{aligned}$$

Quite likely Ramanujan discovered Entry 10 about the same time that Hardy established the aforementioned special case. It is unfortunate that these two great mathematicians had not been able to collaborate ten years earlier than they did, for Ramanujan possessed the more general theorem, while Hardy might have supplied a rigorous proof.

Example 2. *As x tends to ∞,*

$$e^{-x}\sum_{k=1}^{\infty}\frac{x^k\,\mathrm{Log}(k+1)}{k!}=\mathrm{Log}\,x+\frac{1}{2x}+\frac{1}{12x^2}+O\left(\frac{1}{x^3}\right).$$

Proof. Letting $\varphi(x)=\mathrm{Log}(x+1)$ in (10.3) with $M=4$, we deduce that

$$\begin{aligned}e^{-x}\sum_{k=1}^{\infty}\frac{x^k\,\mathrm{Log}(k+1)}{k!}&=\mathrm{Log}(x+1)-\frac{x}{2(x+1)^2}+\frac{x}{3(x+1)^3}\\&\quad-\frac{3x^2}{4(x+1)^4}+O\left(\frac{1}{x^3}\right)\\&=\mathrm{Log}\,x+\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{2x}\left(1-\frac{2}{x}\right)\\&\quad+\frac{1}{3x^2}-\frac{3}{4x^2}+O\left(\frac{1}{x^3}\right),\end{aligned}$$

as x tends to ∞. The desired asymptotic expansion now readily follows.

Ramanujan returns to Example 2 in Section 12 of Chapter 13 where he calculates additional terms of the asymptotic series. Pollak and Shepp [1] have proposed an asymptotic expansion equivalent to that of Example 2.

Example 3.

$$\mathrm{Log}\left(\sum_{k=0}^{\infty}\frac{100^k\varphi(k)}{k!}\right)=100+\mathrm{Log}\left(\frac{\varphi(110)+\varphi(90)}{2}\right)\quad\text{“nearly”.}$$

Proof. We have quoted Ramanujan above, who evidently uses the approximation $\varphi_\infty(x)\sim\varphi(x)$, sets $x=100$, and then replaces $\varphi(100)$ by $\{\varphi(110)+\varphi(90)\}/2$.

Example 4. *Let $\psi(x)=\sum_{k\le x}1/k$. Then as x tends to ∞,*

$$\sum_{k=1}^{\infty}\frac{x^k\psi(k)}{k!}=e^x\left(\mathrm{Log}\,x+\gamma+O\left(\frac{1}{x}\right)\right), \tag{10.25}$$

where γ denotes Euler's constant.

Proof. As x tends to ∞ (see Ayoub's text [1, p. 43]),

$$\psi(x)=\mathrm{Log}\,x+\gamma+O\left(\frac{1}{x}\right). \tag{10.26}$$

Now substitute (10.26) into the left side of (10.25). Then apply (10.3) to $\varphi(x)=\mathrm{Log}\,x$ with $M=1$. The result now easily follows.

An independent proof of Example 4 can be gotten by employing Corollary

2 of Entry 2. We omit the details. Anticipating his work on divergent series in Chapter 6, Ramanujan calls γ (c in his notation) the "constant" of the series $\sum_{k=1}^{\infty} 1/k$.

Entry 11. *Suppose that* $f(x) = \sum_{n=1}^{\infty} A_n x^n/n$ *is analytic for* $|x| < R$. *Define* p_n, $0 \le n < \infty$, *by*

$$\sum_{n=0}^{\infty} p_n x^n = \exp\{f(x)\}, \qquad |x| < R. \tag{11.1}$$

Then

$$np_n = \sum_{k=1}^{n} A_k p_{n-k}, \qquad n \ge 1. \tag{11.2}$$

Proof. Taking the derivative of both sides of (11.1), we find that

$$\sum_{n=1}^{\infty} np_n x^{n-1} = \sum_{j=0}^{\infty} p_j x^j \sum_{k=1}^{\infty} A_k x^{k-1}, \qquad |x| < R.$$

Equating the coefficients of x^{n-1} on both sides above, we obtain the desired recursion formula.

Corollary. Let $\{a_k\}$, $1 \le k < \infty$, *be a sequence of complex numbers such that* $\sum_{k=1}^{\infty} |a_k| < \infty$. *Let* $S_n = \sum_{k=1}^{\infty} a_k^n$, *where n is a positive integer. For* $n \ge 1$, *define* p_n *to be the sum of all products of n distinct terms taken from* $\{a_k\}$. Let $p_0 = 1$. *Then*

$$np_n = \sum_{k=1}^{n} (-1)^{k-1} S_k p_{n-k}, \qquad n \ge 1.$$

Proof. For $|x| < \rho \equiv \inf_n 1/|a_n|$, $a_n \ne 0$,

$$\sum_{n=0}^{\infty} p_n x^n = \prod_{n=1}^{\infty} (1 + a_n x)$$

is analytic and nonzero. Thus, in the notation of Entry 11, for $|x| < \rho$,

$$\sum_{n=1}^{\infty} \frac{A_n}{n} x^n = \sum_{k=1}^{\infty} \operatorname{Log}(1 + a_k x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} S_n}{n} x^n.$$

Hence, $A_n = (-1)^{n-1} S_n$, $n \ge 1$. Substituting this in (11.2), we complete the proof.

In the corollary above, Ramanujan assumes that the sequence $\{a_k\}$ is finite, but this is unnecessary.

For integral r and complex n, define

$$F_r(n) = \sum_{k=0}^{\infty} \frac{(n+k)^{r+k}}{a^k k!}. \tag{12.1}$$

By Stirling's formula (I6), the $(k + 1)$th term of $F_r(n)$ is asymptotic to

$(1/\sqrt{2\pi})k^{r-1/2}(e/a)^k$ as k tends to ∞. Hence, $F_r(n)$ converges for $|a| > e$ and also for $|a| = e$ if $r < -1/2$.

Entry 12. *For r, n, and a as specified above, we have*

$$F_{r+1}(n) = nF_r(n) + \frac{1}{a} F_{r+1}(n+1).$$

Proof. We have

$$F_{r+1}(n) = nF_r(n) + \sum_{k=0}^{\infty} \frac{k(n+k)^{r+k}}{a^k k!}$$
$$= nF_r(n) + \frac{1}{a} \sum_{k=1}^{\infty} \frac{(n+k)^{r+k}}{a^{k-1}(k-1)!},$$

from which the desired recursion follows.

Entries 13 and 14 are concerned with functions of the form

$$\varphi(x) = \sum_{n=0}^{\infty} f_n(x) a^n,$$

where the coefficients are polynomials in x such that

$$f_n(x+y) = \sum_{k=0}^{n} f_k(x) f_{n-k}(y), \qquad 0 \le n < \infty, \tag{13.1}$$

where x and y are arbitrary real numbers. Thus, φ satisfies the relation

$$\varphi(x+y) = \varphi(x)\varphi(y) \tag{13.2}$$

for all values of x and y.

We first prove a general theorem and corollary from which we shall deduce the identities of Entries 13 and 14.

Theorem. *Let* p and q ***be*** constants with $q \neq 0$. *Let* $f_n(x)$, $0 \le n < \infty$, be a ***sequence of polynomials satisfying the difference equation***

$$f_n(x+q) - f_n(x) = qf_{n-1}(x+p), \qquad n \ge 1, \tag{13.3}$$

together with the initial conditions

$$f_0(x) \equiv 1 \tag{13.4}$$

and

$$f_n(0) = 0, \qquad n \ge 1. \tag{13.5}$$

Then $f_n(x)$ satisfies (13.1).

Before commencing the proof, it might be noted that the theorem remains true if the factor q on the right side of (13.3) is replaced by a third arbitrary constant r, $r \neq 0$. For if the solutions of (13.3)–(13.5) are denoted by $f_n(x)$, those solutions under the modified conditions are $(r/q)^n f_n(x)$, $0 \leq n < \infty$. Thus, for the apparent generalization, both sides of (13.1) are merely multiplied by $(r/q)^n$.

Proof of Theorem. We shall induct on n. By (13.4), the relation (13.1) holds for $n = 0$. Suppose that (13.1) is valid for all values of x and y when $0 \leq n \leq m-1$.

We shall first show that (13.1) is true for all values of x when $n = m$ and $y = q$. From (13.3) and (13.5),

$$f_n(q) = qf_{n-1}(p), \qquad n \geq 1. \tag{13.6}$$

By (13.3), (13.6), and the induction hypothesis,

$$\begin{aligned} f_m(x+q) &= f_m(x) + q \sum_{k=0}^{m-1} f_k(x) f_{m-1-k}(p) \\ &= f_m(x) + \sum_{k=0}^{m-1} f_k(x) f_{m-k}(q) \\ &= \sum_{k=0}^{m} f_k(x) f_{m-k}(q), \end{aligned} \tag{13.7}$$

by (13.4).

Next, we shall show that if (13.1) is valid for all values of x when $n = m$ and y has a particular value y_0, then (13.1) also holds for all values of x when $n = m$ and $y = y_0 + q$. By (13.7),

$$\begin{aligned} f_m(x + y_0 + q) &= \sum_{k=0}^{m} f_k(x + y_0) f_{m-k}(q) \\ &= \sum_{k=0}^{m} f_{m-k}(q) \sum_{j=0}^{k} f_j(x) f_{k-j}(y_0), \end{aligned}$$

by our assumptions. Now invert the order of summation and put $r = k - j$ to obtain

$$\begin{aligned} f_m(x + y_0 + q) &= \sum_{j=0}^{m} f_j(x) \sum_{r=0}^{m-j} f_r(y_0) f_{m-j-r}(q) \\ &= \sum_{j=0}^{m} f_j(x) f_{m-j}(y_0 + q), \end{aligned}$$

by the induction hypothesis and by (13.7) when $j = 0$. We have thus shown that (13.1) is valid for all values of x when $n = m$ and y is any positive integral multiple of q. In other words, the polynomial identity (13.1), when $n = m$, is valid for all x and an infinite number of values of y, and so must be valid for all x and all y.

Corollary. *Let p and q be constants. Then the polynomials* $f_0(x) = 1, f_1(x) = x$, *and*

$$f_n(x) = \frac{x}{n!}\prod_{k=1}^{n-1}(x + np - kq), \qquad n \ge 2,$$

satisfy (13.1).

Proof. For $q \neq 0$, the result is obvious for $n = 0$ and follows from the Theorem when $n \ge 1$. For $q = 0$, the result follows by continuity in q.

The theorem above and its corollary are not explicitly stated by Ramanujan in his notebooks. The theorem's proof that we have given was supplied to Wilson by U. S. Haslam-Jones. The theorem and corollary are now part of a general theory developed by Rota and Mullin [1, p. 182]. The polynomials in the corollary were first introduced in the literature by Jensen [2] in 1902 and later by Gould [1], and are essentially what are now called the Gould polynomials. (Consult the papers of Rota, Kahaner, and Odlyzko [1, pp. 733–736] and Roman and Rota [1, p. 115].) See Gould's papers [1]–[6], and a paper of Carlitz [4] for several formulas and the context in which these polynomials arise.

Entry 13. *Let* $f(n) = nF_{-1}(n)$, *where* F_{-1} *is defined by* (12.1). *Assume that a is real, with* $|a| \ge e$. *Then there exists a positive real number x satisfying the relation* $x = a \operatorname{Log} x$ *such that for any real number n,* $x^n = f(n)$.

Proof. By the corollary with $p = 1$ and $q = 0$ and by (13.2), $f(m)f(n) = f(m+n)$, where m and n are arbitrary real numbers. Hence, if n is any positive integer, $f(n) = x^n$, where $x = f(1)$. This relation may be extended to negative integers n by using the equality $f(n)f(-n) = f(0) = 1$. It can further be extended to all rational numbers r/s upon noting that $\{f(r/s)\}^s = f(r) = x^r$. For $|a| \ge e$, $f(n)$ converges uniformly on any compact interval in the variable n. Hence, $f(n)$ is continuous for all n. It follows that $f(n) = x^n$ for all real values of n. Hence, for $|a| \ge e$,

$$f'(n) = x^n \operatorname{Log} x = \sum_{k=1}^{\infty}\frac{(n+k)^{k-1}}{a^k k!} + \sum_{k=1}^{\infty}\frac{n(k-1)(n+k)^{k-2}}{a^k k!},$$

since both of these series converge uniformly on any compact interval in n. Thus,

$$f'(0) = \operatorname{Log} x = \sum_{k=1}^{\infty}\frac{k^{k-2}}{a^k(k-1)!} = \frac{f(1)}{a} = \frac{x}{a}.$$

This completes the proof.

Let a now be complex and consider the relation $x = a \operatorname{Log} x$, where x is to be regarded as a function of a. By considering, for example, the graph of

$x/\mathrm{Log}\, x$ for real values of x, we see that for a > e there are two branches x_1 and x_2 of the function x that have real values. Thus, a = e is a branch point. One branch, say x_1, decreases from e to 1 as a increases from e to $+\infty$. The other branch x_2 increases from e to $+\infty$ as a increases from e to $+\infty$. Since $f(1)$ tends to 1 as a tends to ∞, it follows that, for $a > e$, $f(1)$ defines a branch of the function $x(a)$ that is real and lies between 1 and e. Entry 13 thus shows that $f(n) = x_1^n$.

Corollary. *Let z be an arbitrary complex number, and suppose that w is any complex number such that* $|e^{w-1}| \geq |w|$. *Then*

$$e^z = \sum_{k=0}^{\infty} \frac{z(z+kw)^{k-1}e^{-kw}}{k!} \tag{13.8}$$

Proof. First suppose that w is real. Apply Entry 13 with $x = e^w$, and so $a = e^w/w$. Then for any real number n,

$$f(n) = e^{nw} = \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}w^k e^{-kw}}{k!}.$$

Letting $z = nw$, we deduce (13.8) for real z and real w with $|w| \leq |e^{w-1}|$. By analytic continuation in each of the variables w and z, we complete the proof.

For brevity, we shall now define $c_0(n) = 1$, $c_1(n) = n$, and

$$c_k(n) = n\prod_{j=1}^{k-1}(n + kp - jq), \quad k \geq 2. \tag{14.1}$$

Define, for complex a,

$$\varphi(n) = \sum_{k=0}^{\infty} \frac{c_k(n)a^k}{k!}. \tag{14.2}$$

If $p = q = 0$, $\varphi(n) = e^{an}$. If $p \neq 0$ but $q = 0$, then $\varphi(n) = f(n/p)$, where f is the function defined in Entry 13 but with a replaced by $1/ap$. If $p = 0$ and $q \neq 0$, then $\varphi(n) = (1 + aq)^{n/q}$. If $p = q \neq 0$, then $\varphi(n) = (1 - ap)^{-n/p}$. Thus, in the sequel we may suppose that none of the parameters, p, q, and $p - q$ is equal to 0. Furthermore, without loss of generality, we may assume that p and q are positive, for, in a more explicit notation,

$$\begin{aligned}\varphi(n) &= \varphi(n; p, q, a) = \varphi(-n; -p, -q, -a)\\ &= \varphi(n; p-q, -q, a) = \varphi(-n; q-p, q, -a).\end{aligned}$$

Now, by Stirling's formula (I6), as k tends to ∞,

$$\left|\frac{c_k(n)}{k!}\right| = \left|\frac{nq^{k-1}\Gamma\left(\dfrac{n+kp}{q}\right)}{k!\,\Gamma\left(\dfrac{n+kp}{q} - k + 1\right)}\right| \sim ck^{-3/2}p^{pk/q}|p-q|^{-k(p-q)/q},$$

where the constant c depends upon p, q, and n but not k. Thus, $\varphi(n)$ converges for

$$|a| \leq p^{-p/q}|p-q|^{(p-q)/q}. \tag{14.3}$$

Entry 14. ***Let φ be defined by* (14.2) *and let p and q be as specified above. If x is a certain root of the equation***

$$aqx^p - x^q + 1 = 0, \tag{14.4}$$

then $\varphi(n) = x^n$ for every real number n.

Proof. By the same type of argument as that in the proof of Entry 13,

$$\varphi(n) = x^n, \tag{14.5}$$

where $\varphi(1) = x$ and n is any real number. Next, by a direct calculation,

$$\frac{1}{k!}c_k(q) - \frac{q}{(k-1)!}c_{k-1}(p) = 0.$$

Hence, since $c_0(n) = 1$,

$$\varphi(q) - aq\varphi(p) = 1.$$

In other words, by (14.5), x satisfies (14.4), and the proof is complete.

Note that, by (14.2), $x=x(a)$ tends to 1 as a tends to 0.

Corollary 1. ***Let n be real and suppose that $|a| \leq 1/4$. Then***

$$\left(\frac{2}{1+\sqrt{1-4a}}\right)^n = 1 + na + n\sum_{k=2}^{\infty}\frac{\Gamma(n+2k)a^k}{\Gamma(n+k+1)k!}.$$

Proof. In Entry 14, let $p = 2q$. The root of (14.4) which tends to unity as a tends to 0 is given by

$$x^q = \frac{1-\sqrt{1-4aq}}{2aq} = \frac{2}{1+\sqrt{1-4aq}}.$$

Thus, by Entry 14 and (14.1),

$$\left(\frac{2}{1+\sqrt{1-4aq}}\right)^n = \varphi(nq) = 1 + nqa + nq\sum_{k=2}^{\infty}\left\{\prod_{j=k+1}^{2k-1}(nq+jq)\right\}\frac{a^k}{k!},$$

where $|a| < 1/(4q)$, by (14.3). Setting $q = 1$ in the equalities above we complete the proof.

Corollary 2. ***Let n be real and assume that $|a| \leq 1$. Then***

$$(a+\sqrt{1+a^2})^n = 1 + na + \sum_{k=2}^{\infty}\frac{b_k(n)a^k}{k!},$$

where, for $k \geq 2$,

$$b_k(n) = \begin{cases} n^2(n^2-2^2)(n^2-4^2)\cdots(n^2-(k-2)^2), & \text{if } k \text{ is even,} \\ n(n^2-1^2)(n^2-3^2)\cdots(n^2-(k-2)^2), & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let $q = 2p$ in Entry 14. The root of (14.4) which tends to 1 as a approaches 0 is given by $x^p = ap + \sqrt{a^2p^2+1}$. Hence, by Entry 14,

$$(ap + \sqrt{a^2p^2+1})^n = \varphi(np) = 1 + npa + \sum_{k=2}^{\infty} \frac{c_k(np)a^k}{k!},$$

where $c_k(np)$ is given by (14. 1), and where $|a| \leq 1/p$ by (14.3). Now let $p = 1$ and $q = 2$ in the above equalities. Then it is not very hard to see that $c_k(n) = b_k(n)$, and the proof is complete.

Entries 13 and 14 have a long history. Entry 14 was first established by Lambert [1, pp. 38–40] in a paper published in 1758. In 1770, Lagrange [1] published a proof of the celebrated "Lagrange inversion formula." As an application, he derived Entry 14 [1, pp. 53–56]. Entries 13 and 14 appear as problems illustrating the Lagrange inversion formula in the text of Pólya and Szegö [1, pp. 145–146]. In 1779, in a paper stimulated by the work of Lambert, Euler [4], [6] proved both Entries 13 and 14. Entry 13 actually appears in a paper published by Euler [3] one year earlier, but no proof is given. Rothe [1] rediscovered the special case $q = 1$ of Entry 14 in his dissertation published in 1793. Entries 13 and 14 also follow from Abel's [1], [2, pp. 102–103] generalization of the binomial theorem and are sometimes attributed to him. Entry 13 was rediscovered in 1844 by Eisenstein [1], [2, pp. 122–125] who was apparently unaware of earlier work. Other proofs of Entry 13 or equivalent formulations have been given by Wittstein [1] in 1845, Woepcke [1] in 1851, Seidel [1] in 1873, and Jensen [2] in 1902. The result is also found in Gould's paper [2, p. 412]. A similar theorem of a more general type has also been established by Gould [4, Theorem 7]. Entry 14 is similar to further results of Gould [1, p. 85], [4, Theorem 1]. Entry 14 has also been generalized in a different direction; solutions of certain algebraic equations can be represented by hypergeometric series. Further references can be found in Birkeland's paper [1]. Hardy [20, p. 194] refers to Ramanujan's work on (14.4). Moreover, Ramanujan discusses (14.4) in his quarterly reports. The corollary of Entry 13 is essentially a reformulation of an exercise in Bromwich's book [1, p. 160]. (See also p. 195 of Bromwich's text.) The aforementioned corollary is also derived by means of the Lagrange inversion formula in Chaundy's text [1, p. 409] and Carmichael's paper [1]. An application of this corollary has been given by Rogers [1]. Jackson [1] has found a q-analogue of this corollary as well as of Abel's theorem and related results. Gould [7] has compiled an extensive bibliography of papers related to Entries 13 and 14, the aforementioned convolution theorem of Abel, and similar results. Finally, an article by Knoebel [1] contains many references to Entry 13 and allied results.

Entry 15. *Define u,* $0 < u < 1$*, by*

$$u - \operatorname{Log} u = 1 + \frac{x^2}{2}, \tag{15.1}$$

where x is real. Then

$$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} e^{-k(1+x^2/2)} = ue^{1+x^2/2} \tag{15.2}$$

Furthermore, for sufficiently small positive x,

$$u = \sum_{k=0}^{\infty} b_k x^k,$$

where $b_0 = 1$, $b_1 = -1$, $b_2 = 1/3$, $b_3 = -1/36$, $b_4 = -1/270$, and, ***in general, the coefficients*** b_k ***are found successively by substituting into the identity***

$$\tfrac{1}{2}x^2 = \sum_{j=2}^{\infty} \frac{(1-u)^j}{j}.$$

Proof. In Entry 13, let $x = e^u$ and $a = e^u/u$, so $a = \exp(1 + x^2/2)$ by (15.1). We then find that

$$f(1) = e^u = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} e^{-k(1+x^2/2)},$$

from which (15.2) follows.

Define $F(u) = 2(u - 1 - \operatorname{Log} u) = x^2$. Note that F is analytic in a neighborhood N of $u = 1$ and that $F'(1) = 0$ and $F''(1) \neq 0$. For $u \in N$, we may then write $F(u) = G(u)^2 = x^2$, where $G(u)$ is analytic and one-to-one on N, and where, say, $x = G(u)$ and $G'(1) < 0$. Thus, there exists an analytic inverse of $G(u) = x$ in a neighborhood of $x = 0$ of the form

$$u = G^{-1}(x) = \sum_{k=0}^{\infty} b_k x^k.$$

When $0 < u < 1$, the equalities above hold with $x > 0$, since $G'(1) < 0$. We have $b_0 = 1$ and b_1 is negative. By (15.1) and Taylor's theorem,

$$\tfrac{1}{2}x^2 = u - 1 - \operatorname{Log} u = \sum_{j=2}^{\infty} \frac{(1-u)^j}{j},$$

and so the coefficients b_k may be calculated as indicated.

Example 1. ***Let m be real and let*** $0 < n \leq 2$***. Then***

$$2^m = \sum_{k=0}^{\infty} \frac{m\Gamma(m+kn)}{\Gamma(m+kn-k+1)2^{kn}k!}.$$

This example is highly interesting. Ramanujan, in fact, claims the result is true for $0 < n < \infty$. The series does converge for $0 < n < \infty$. However, it converges to a different value for $n > 2$. In the proof below, we shall prove this last fact as well.

Proof. In Entry 14, replace n by m, let $q = 1$, replace p by n, and let $a = 2^{-n}$, where $0 < n < \infty$. We find that if x is a certain root of

$$f(x) \equiv 2^{-n}x^n - x + 1 = 0, \tag{15.3}$$

then

$$x^m = \sum_{k=0}^{\infty} \frac{m\Gamma(m + kn)}{\Gamma(m + kn - k + 1)2^{kn}k!}, \tag{15.4}$$

provided the series converges. By (14.3), the series above converges if

$$2^{-n} \leq n^{-n}|n - 1|^{n-1}, \tag{15.5}$$

for $n \neq 1$. By the remarks made prior to Entry 14, the series in (15.4) converges for $n = 1$, in which case (15.5) would be interpreted as $2^{-1} \leq 1$. We now show that (15.5) holds for $0 < n < \infty$. Letting $g(x) = (x/2)^x|1 - x|^{1-x}$, we want to show that $g(x) \leq 1$ for $x \geq 0$. By elementary calculus, we find that $g(x)$ decreases for $0 < x < 2/3$ from the value $g(0) = 1$. On $2/3 < x < 2$, g increases to the value $g(2) = 1$. For $x > 2$, g decreases. Thus, $g(x) \leq 1$ for $x \geqq 0$, and (15.5) is valid for $0 < n < \infty$.

Now, obviously, $x = 2$ is a root of (15.3). Since $f'(x) = n2^{-n}x^{n-1} - 1$, we see that there is a unique positive value $x = \xi$ such that $f'(\xi) = 0$. Hence, (15.3) has at most one positive root in addition to the root $x = 2$. If $0 < n \leq 1$, $f(0)$ is positive while $f(+\infty)$ is negative. Hence, $x = 2$ is the only positive root of (15.3). Thus, Example 1 is established for $0 \leq n < 1$. If $n > 1$, both $f(0)$ and $f(+\infty)$ are positive. Thus, in addition to the root $x = 2$, (15.3) has another (not necessarily distinct) positive root $x = \alpha$, and clearly $\alpha > 1$. Now $\alpha = 2$ if and only if $f'(2) = 0$, which happens only when $n = 2$. Thus, Example 1 is valid for $n = 2$. Observe that $f'(2)$ has the same sign as $n - 2$. Thus, $\alpha > 2$ if $1 < n < 2$, but $\alpha < 2$ if $n > 2$. Also, as n tends to $1+$, α tends to ∞; as n tends to 2, α tends to 2.

To complete the argument, we only need to show that the series in (15.4) converges uniformly in $1 \leq n \leq 2$ for one particular value of $m \neq 0$; the sum is then a continuous function of n, and so $x = 2$ for $1 < n < 2$. Choose $m = -1/2$. By (15.5) and Stirling's formula (16), there exists a constant K, independent of k and n, such that

$$\left| \frac{\Gamma(-\frac{1}{2} + kn)}{\Gamma(kn - k + \frac{1}{2})2^{kn}k!} \right| \leq Kk^{-3/2}.$$

Hence, the series in (15.4) converges uniformly in n on any interval in $[1, \infty)$ when $m = -1/2$. This completes the proof that (15.4) holds for $x = 2$ and any real value of m when $0 < n \leq 2$.

Lastly, we shall show that for $n > 2$, (15.4) is valid for $x = \alpha < 2$. By the argument above, the series in (15.4) converges uniformly for $2 \leq n < \infty$, and so it is sufficient to prove the assertion for one particular value of n greater than 2. We shall choose $m = 1$ and $n = 3$. Then from (15.3) we see that $a = \sqrt{5} - 1$. We therefore must prove that

$$\sum_{k=0}^{\infty} \frac{\Gamma(3k+1)}{\Gamma(2k+2)8^k k!} = \sqrt{5} - 1. \tag{15.6}$$

Now since this sum is known to be equal to 2 or $\sqrt{5} - 1$, it suffices to show that this sum is less than 2. Let

$$a_k = \frac{\Gamma(3k+1)}{\Gamma(2k+2)8^k k!}$$

Then for $k \geq 2$,

$$\frac{a_{k+1}}{a_k} = \frac{3(3k+1)(3k+2)}{16(k+1)(2k+3)} = \frac{27}{32} - \frac{81k+69}{32(k+1)(2k+3)} < \frac{27}{32}.$$

Hence,

$$\sum_{k=0}^{\infty} a_k < 1 + \frac{1}{8} + \frac{3}{64}\sum_{k=0}^{\infty}\left(\frac{27}{32}\right)^k = 1 + \frac{1}{8} + \frac{3}{10} < 2.$$

This establishes (15.6). Thus, we have shown that the equality in Example 1 is valid for $0 < n \leq 2$ and invalid for $n > 2$.

In connection with all of the examples below, it should be kept in mind that in the proofs the relevant root X of the equation $X = A \operatorname{Log} X$, $A \geq e$, as was emphasized in the remarks made after Entry 13, is that root which lies between 1 and e. If X is a root of $X = -A \operatorname{Log} X$, $A \geq e$, then there is no ambiguity, as the root, which is between 0 and 1, is unique.

Example 2. *Let a be positive and suppose that m, n, and p are real and* nonzero. *Define the positive real number x by the relation* $(\operatorname{Log} x)^m = ax^n$. *Then, for $a \leq |m/en|^m$,*

$$x^p = p\sum_{k=0}^{\infty} \frac{(mp+nk)^{k-1}a^{k/m}}{m^{k-1}k!}.$$

Proof. Define $y > 0$ by $x^n = y^n$. A short calculation gives

$$\frac{\operatorname{Log} y}{y} = \frac{n}{m}a^{1/m}.$$

We now apply Entry 13 with x replaced by y and a replaced by $(m/n)a^{-1/m}$. For $|m/n|a^{-1/m} \geq e$, we then have

$$x^p = y^{mp/n} = \sum_{k=0}^{\infty} \frac{(mp/n)(mp/n+k)^{k-1}}{(ma^{-1/m}/n)^k k!},$$

from which the desired result follows.

Examples 3(i)–3(viii) and 4(i)–4(iv) arise from Entry 13 by suitable changes of variables. Example 3(i) is essentially the same as Entry 13 except that x is now defined in either of two ways by x = $\pm$ a Log x. For Example 3(ii), a is replaced by $\pm a$/Log a in Entry 13. The case x = 1 in the first equality of Example 3(iii) below is a problem posed by Newman [1]. This problem can also be deduced from the corollary to Entry 13.

Example 3(iii). ***Let a be** real **with** $a \leq e$ **und dejne the reul number x by either of the two equulities** $x = ae^{\pm x}$. **Then***

$$e^x = \sum_{k=O}^{\infty} \frac{(k+1)^{k-1}a^k}{k!} \quad \textit{und} \quad e^{-x} = \sum_{k=O}^{\infty} \frac{(k+1)^{k-1}(-a)^k}{k!},$$

respectively.

Proof. Let x = Log y. Then we have, respectively, y/Log y = $1/a$ and y^{-1}/Log(y-‘) = $1/a$. Now apply Entry 13 with a replaced by $1/a$ and $-1/a$, respectively.

Example 3(iv) follows from Entry 13 upon replacing a by $\pm$ 1/Log a.

Example 3(v). ***Let a be positive with*** Log $a| \leq e$. ***Define a positive real number x by either of the two relations*** $x^{\pm x} = a$. ***Then***

$$\frac{1}{x} = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}(\mp \operatorname{Log} a)^k}{k!}.$$

Proof. Let x = $1/y$. Then it follows that $\mp$(Log y)/y = Log a. Now apply Entry 13 with x replaced by y and a replaced by $\mp$1/Log a.

Example 3(vi) is identical to Example 3(iii) except that the relation x = $ae^{\pm x}$ has been replaced by x = $ae^{\mp x}$ in the second notebook.

Example 3(vii). ***Let a be** real **und** define **the** real **number x by either of the relations** $e^x \pm x = a$. **Then, respectively,*** *if* $a \leq -1$,

$$e^{-e^x} = \sum_{k=O}^{\infty} \frac{(k+1)^{k-1}(-1)^k e^{ak}}{k!},$$

und *if* $a \geq 1$,

$$e^{e^x} = \sum_{k=O}^{\infty} \frac{(k+1)^{k-1}e^{-ak}}{k!},$$

Proof. Let x = Log Log y. Then e^a = y(Log $y)^{\pm 1}$. In the former case Log($1/y$)/($1/y$) = e^a, and in the latter case y/Log y = e^a. Now apply Entry 13. In the former case x is replaced by $1/y$ and a is replaced by $-e^{-a}$; in the latter case x is replaced by y and a by e^a.

Example 3(viii) is the same as Example 3(vii), except that x has been replaced by Log x.

Example 4(i). ***Let x > 0 and define v by v = x”. Then for*** Log x $\leq 1/e$,

$$v = \sum_{k=O}^{\infty} \frac{(k+1)^{k-1}(\text{Log } x)^k}{k!}.$$

Proof. In Entry 13, replace ***a*** by 1/Log x. Then x is replaced by v.

For v Ramanujan writes $x^{x^{X^{\&c}}}$. It is curious that Eisenstein [1] used this same notation. In Examples 4(ii)–4(iv), similar notations are employed by Ramanujan.

If we interpret $x^{x^{x^{\cdot}}}$ as the limit of the sequence, x, x^x, x$^{(x^x)}$, . . . , then $x^{X^{x^{\cdot}}}$ converges for $e^{-e} \leq$ x $\leq e^{1/e}$. This fact appears as an exercise in several texts, including those of Apostol [1, Exer. 12-7, p. 383], Bromwich [1, Exer. 11, p. 23], Knopp [1, Exer. 23, p. 108], Bender and Orszag [1, Exer. 4.2, p. 196], and Francis and Littlewood [1, pp. 10, 38, 39]. This exercise has also been posed as a journal problem by several people including Lense [1] and Ogilvy [1]. A paper by Andrews and Lacher [1] examines the convergence in great detail.

The more general problem concerning the convergence of

$$x_1^{x_2^{x_3}}$$

has been studied by many, including Barrow [1], Creutz and Sternheimer [1], and Shell [1]. The latter paper gives a synopsis of earlier results. For many references on the aforementioned and related problems, consult the comprehensive survey paper by Knoebel [1].

Example 4(ii) is the same as Example 3(vii) but with x and ***a*** replaced by v and $\pm$ x, respectively. Example 4(iii) is simply a reformulation of Entry 13 with x replaced by e” and ***a*** replaced by x. Example 4(iv) is another version of Example 4(ii), but with x replaced by $-x$ in the former equality and v replaced by $-v$ in the latter equality; in other words, x and v satisfy either of the relations $v = \pm\,\text{Log}(x$ + u).

Ramanujan next attempts to generalize Entry 13 by considering the functions $\varphi_r(n)$ defined by the equation

$$x^n \varphi_r(n) = F_r(n), \tag{16.1}$$

where $x = a \operatorname{Log} x$, $F_r(n)$ is defined in (12.1), and where a is specified prior to Entry 12. Thus, $\varphi_{-1}(n) = 1/n$.

Entry 16(a). *If r is an integer and n is any complex number, then*

$$n\varphi_r(n) = \varphi_{r+1}(n) - (\operatorname{Log} x)\varphi_{r+1}(n+1). \tag{16.2}$$

Proof. Using (16.1) and the relation $x/a = \operatorname{Log} x$ in Entry 12, we readily deduce (16.2).

Putting $r = -2$ in (16.2), we find that

$$\varphi_{-2}(n) = \frac{1}{n^2} - \frac{\operatorname{Log} x}{n(n+1)} = \frac{1 - \operatorname{Log} x}{n(n+1)} + \frac{1}{n^2(n+1)}.$$

Letting $r = -3$ in (16.2), we get

$$\varphi_{-3}(n) = \frac{1 - \operatorname{Log} x}{n^2(n+1)} + \frac{1}{n^3(n+1)} - \frac{\operatorname{Log} x}{n}\left\{\frac{1 - \operatorname{Log} x}{(n+1)(n+2)} + \frac{1}{(n+1)^2(n+2)}\right\}$$

$$= \frac{(1 - \operatorname{Log} x)^2}{n(n+1)(n+2)} + \frac{(3n+2)(1 - \operatorname{Log} x)}{n^2(n+1)^2(n+2)} + \frac{3n+2}{n^3(n+1)^2(n+2)}.$$

Both of these formulas for $\varphi_{-2}(n)$ and $\varphi_{-3}(n)$ were given by Ramanujan. It is clear from the recursion formula (16.2) that $\varphi_{-k}(n)$, $k \geq 1$, is a polynomial in $\operatorname{Log} x$ of degree $k - 1$ with the coefficients being rational functions of n.

We now turn to the calculation of $\varphi_r(n)$ when r is nonnegative. Let $x = e^u$, and so $a = e^u/u$. Putting $\varphi_r(n) = g_r(n, u) = g_r(u)$, we find from (16.1) that

$$g_r(u) = \sum_{k=0}^{\infty} \frac{(n+k)^{r+k} e^{-u(n+k)} u^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(n+k)^{r+k+j}(-1)^j u^{k+j}}{k!\, j!}.$$

We first calculate $g_0(u)$. Using the latter representation above, we find that

$$g_0^{(m)}(u) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(n+k)^{k+j}(-1)^j (k+j)!\, u^{k+j-m}}{k!\, j!\, (k+j-m)!}, \qquad m \geq 0,$$

and so $g_0^{(m)}(0) = S(m, m)$, where

$$S(m, m) = \sum_{k=0}^{m} (-1)^{k+m} \binom{m}{k} (n+k)^m.$$

It is well known (see, e.g., Hansen's tables [1, p. 134]) that $S(m, m) = m!$. Thus,

$$g_0(u) = \sum_{m=0}^{\infty} u^m = \frac{1}{1-u}. \tag{16.3}$$

Next, we shall show, by induction on r, that

$$g_r(u) = \sum_{k=1}^{r+1} \frac{\psi_k(r, n)}{(1-u)^{r+k}}, \qquad r \geq 0, \tag{16.4}$$

where $\psi_k(r, n)$, $1 \leq k \leq r+1$, is independent of u. By (16.3), (16.4) is valid for $r = 0$. A direct calculation shows that, for $r \geq 1$,

$$\begin{aligned}(1-u)g_r(u) &= ng_{r-1}(u) + ug'_{r-1}(u)\\ &= ng_{r-1}(u) - (1-u)g'_{r-1}(u) + g'_{r-1}(u).\end{aligned}$$

Using the induction hypothesis, we thus find from (16.4) that

$$\begin{aligned}(1-u)g_r(u) = n \sum_{k=1}^{r} \frac{\psi_k(r-1, n)}{(1-u)^{r+k-1}} &- \sum_{k=1}^{r} \frac{(r+k-1)\psi_k(r-1, n)}{(1-u)^{r+k-1}}\\ &+ \sum_{k=1}^{r} \frac{(r+k-1)\psi_k(r-1, n)}{(1-u)^{r+k}},\end{aligned}$$

or

$$g_r(u) = \sum_{k=1}^{r+1} \frac{(n-r-k+1)\psi_k(r-1, n) + (r+k-2)\psi_{k-1}(r-1, n)}{(1-u)^{r+k}},$$

where we define $\psi_0(r-1, n) = 0 = \psi_{r+1}(r-1, n)$. Hence, (16.4) is established, and moreover we have proven the recursion formula

$$\psi_k(r, n) = (n-r-k+1)\psi_k(r-1, n) + (r+k-2)\psi_{k-1}(r-1, n), \tag{16.5}$$

where $1 \leq k \leq r+1$.

Entry 16(b). *Let r and t be integers such that $1 \leq t \leq r+2$. Then*

$$\psi_t(r+1, n) = (n-1)\psi_t(r, n-1) + \psi_{t-1}(r+1, n) - \psi_{t-1}(r+1, n-1),$$

where $\psi_t(r, n) = 0$ if $t \notin \{k\colon 1 \leq k \leq r+1\}$.

Proof. Employing (16.4) in (16.2), we obtain the identity

$$\begin{aligned}n \sum_{k=1}^{r+1} \frac{\psi_k(r, n)}{(1-\operatorname{Log} x)^{r+k}} = \sum_{k=1}^{r+2} &\frac{\psi_k(r+1, n)}{(1-\operatorname{Log} x)^{r+k+1}}\\ &+ \{(1-\operatorname{Log} x) - 1\} \sum_{k=1}^{r+2} \frac{\psi_k(r+1, n+1)}{(1-\operatorname{Log} x)^{r+k+1}}.\end{aligned}$$

Equating coefficients of $(1-\operatorname{Log} x)^{-r-t}$, $1 \leq t \leq r+2$, on both sides and replacing n by $n-1$, we deduce the desired recursion formula.

Using either of the recursion formulas given in (16.5) and Entry 16(b) together with the value $\psi_1(0, n) = 1$ from (16.3), we can successively calculate the coefficients of $\varphi_1(n)$, $\varphi_2(n)$, $\varphi_3(n)$, ... We thus find that

$$\varphi_1(n) = \frac{n-1}{(1-\operatorname{Log} x)^2} + \frac{1}{(1-\operatorname{Log} x)^3},$$

$$\varphi_2(n) = \frac{(n-1)(n-2)}{(1-\operatorname{Log} x)^3} + \frac{(n-1)(n-2)\left\{\frac{1}{n-2} + \frac{2}{n-1}\right\}}{(1-\operatorname{Log} x)^4} + \frac{3}{(1-\operatorname{Log} x)^5},$$

$$\varphi_3(n) = \frac{(n-1)(n-2)(n-3)}{(1-\operatorname{Log} x)^4} + \frac{(n-1)(n-2)(n-3)\left\{\frac{1}{n-3} + \frac{2}{n-2} + \frac{3}{n-1}\right\}}{(1-\operatorname{Log} x)^5}$$

$$+ \frac{15n-35}{(1-\operatorname{Log} x)^6} + \frac{15}{(1-\operatorname{Log} x)^7},$$

all of which are given by Ramanujan.

Corollary 1. *Let x be complex. If $-\rho < n < 1$, where ρ is the unique real root of $ye^{y+1} = 1$, then*

$$\frac{e^x}{1-n} = \sum_{k=0}^{\infty} \frac{(x+kn)^k}{k!\, e^{kn}}. \tag{16.6}$$

If $n > 1$, then

$$\frac{e^{mx/n}}{1-m} = \sum_{k=0}^{\infty} \frac{(x+kn)^k}{k!\, e^{kn}}, \tag{16.7}$$

where $0 < m < 1$ and $e^m/m = e^n/n$.

Proof. By (12.1), (16.1), and (16.3), for any y and for $|a| > e$,

$$F_0(y) = \sum_{k=0}^{\infty} \frac{(y+k)^k}{a^k k!} = \frac{t^y}{1-\operatorname{Log} t}, \tag{16.8}$$

where $t = a \operatorname{Log} t$. Recall that if $a > e$, then t is that root of this equation which lies between 1 and e; if $a < -e$, then t denotes the unique real root of this equation.

When $n = 0$, (16.6) reduces to the Maclaurin series for e^x. Let $a = e^n/n$, where $-\rho < n < 1$, $n \neq 0$. In this case, $|a| > e$, and the appropriate root t is equal to e^n. Putting $y = x/n$, we find that (16.8) reduces to (16.6).

If $n = 1$, the series in (16.6) diverges, but if $n > 1$ it converges. In the latter case, let $a = e^m/m$, where m is defined in the hypotheses. Then by (16.6),

$$\frac{e^{my}}{1-m} = \sum_{k=0}^{\infty} \frac{(y+k)^k}{(e^m/m)^k k!} = \sum_{k=0}^{\infty} \frac{(y+k)^k}{(e^n/n)^k k!}$$
$$= \sum_{k=0}^{\infty} \frac{(yn+kn)^k}{k!\, e^{kn}}.$$

Thus, (16.7) readily follows.

As indicated in the proof, this very interesting corollary is a generalization of the very familiar Maclaurin series for e^x. Characteristically, Ramanujan states no conditions on n for (16.6) to hold. By Newton's method, it may readily be shown that $\rho = 0.27846454\ldots$. The second part of Corollary 1, namely (16.7), is not given by Ramanujan. Equality (16.6) was apparently first established by Jensen [2]. Other proofs have been given by Duparc, Lekkerkerker, and Peremans [1] and by Gould [3]. Carlitz [2] has employed this corollary in establishing the orthogonality of a certain set of polynomials.

Corollary 2. *For each nonnegative integer r,*

$$n^r = \lim_{a \to \infty} \varphi_r(n) = \sum_{k=1}^{r+1} \psi_k(r, n).$$

Proof. The first equality above follows immediately from the definitions of $F_r(n)$ and $\varphi_r(n)$ given in (12.1) and (16.1), respectively. The second equality follows from the definition of $\psi_k(r, n)$ in (16.4).

Next, fix $a > 1/e$. For real h, define $x > 0$ by the relation

$$x^x = a^a e^h. \tag{17.1}$$

Then $x \operatorname{Log} x = a \operatorname{Log} a + h$ and $(1 + \operatorname{Log} x)\, dx/dh = 1$, i.e.,

$$(x + a \operatorname{Log} a + h)\frac{dx}{dh} = x. \tag{17.2}$$

At $h = 0$, $x = a$ and

$$\frac{dx}{dh} = \frac{1}{1 + \operatorname{Log} a} \equiv n. \tag{17.3}$$

Since $h = h(x)$ extends to a one-to-one analytic function in a neighborhood of a, there is an analytic inverse $x = x(h)$ in a neighborhood of the origin. Thus we have an expansion of the form

$$\frac{x-a}{a} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{A_k}{k!} \left(\frac{h}{a}\right)^k, \tag{17.4}$$

where $|h|$ is sufficiently small.

Entry 17. *For $r \geq 2$, we have*

$$A_r = n(r-2)A_{r-1} + n\sum_{k=1}^{r-1}\binom{r-1}{k}A_kA_{r-k}. \tag{17.5}$$

Proof. Substituting (17.4) in the differential equation (17.2), we obtain the identity

$$\left(\operatorname{Log} a + \frac{h}{a}\right)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{(k-1)!}\left(\frac{h}{a}\right)^{k-1}$$
$$= \left\{1 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{k!}\left(\frac{h}{a}\right)^k\right\}\left\{1 - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{(k-1)!}\left(\frac{h}{a}\right)^{k-1}\right\},$$

or, by (17.3),

$$\left(1 + n\frac{h}{a}\right)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{(k-1)!}\left(\frac{h}{a}\right)^{k-1}$$
$$= n + n\sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{k!}\left(\frac{h}{a}\right)^k\left\{1 - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{(k-1)!}\left(\frac{h}{a}\right)^{k-1}\right\}.$$

Equating coefficients of $(h/a)^{r-1}$, $r \geq 2$, we obtain

$$\frac{A_r}{(r-1)!} - n\frac{A_{r-1}}{(r-2)!} = n\left\{-\frac{A_{r-1}}{(r-1)!} + \sum_{k=1}^{r-1}\frac{A_k}{k!}\frac{A_{r-k}}{(r-1-k)!}\right\}.$$

The recurrence relation (17.5) now easily follows.

In Ramanujan's version of (17.5) the terms with $k=j$ and $k=r-j$, $1 \leq j < r/2$, are combined. However, he has incorrectly written the last term. On page 36, line 4, multiply $A_{(r-1)/2}$ by $A_{(r+1)/2}$ and square $A_{r/2}$.

We have seen that $dx/dh = 1/(1 + \operatorname{Log} x) \equiv N$ and

$$A_1 = \left.\frac{dx}{dh}\right|_{h=0} = N\bigg|_{x=a} = n. \tag{17.6}$$

In general, it follows from (17.4) that for $r \geq 1$,

$$A_r = (-a)^{r-1}\left.\frac{d^rx}{dh^r}\right|_{h=0}. \tag{17.7}$$

Inducting on r, we find that there are numbers, $a(r, k)$ for which

$$\sum_{k=0}^{r-2} a(r, k)N^{2r-k-1} = (-x)^{r-1}\frac{d^rx}{dh^r}, \qquad r \geq 2. \tag{17.8}$$

Differentiating both sides of (17.8) with respect to h and comparing coefficients of N^{2r-k+1}, we obtain the following recursion formula given by

Ramanujan:

$$a(r+1, k) = (r-1)a(r, k-1) + (2r-k-1)a(r, k), \qquad (17.9)$$

where $r \geq 2$, $0 \leq k \leq r-1$, and $a(r, k)$ is defined to be 0 when $k < 0$ or $k > r - 2$. Setting $h = 0$ in (17.8) and using (17.7), we have

$$A_r = \sum_{k=0}^{r-2} a(r, k) n^{2r-k-1}, \qquad r \geq 2. \qquad (17.10)$$

From (17.6), (17.9), and (17.10), Ramanujan has calculated A_r $(1 \leq r \leq 7)$ as follows:

$$A_1 = n,$$
$$A_2 = n^3,$$
$$A_3 = 3n^5 + n^4,$$
$$A_4 = 15n^7 + 10n^6 + 2n^5,$$
$$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6,$$
$$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7,$$
$$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8.$$

Example 1. *For* $n = 1$ *and* $r \geq 2$,

$$\sum_{k=0}^{r-2} a(r, k) = A_r = (r-1)^{r-1}.$$

Proof. The first equality follows from setting $n = 1$ in (17.6). In (17.1), let $a = 1$ and $x = 1/y$. We then easily find that $t/\text{Log } y = -1/h$. Now apply Entry 13 with $n = -1$, x replaced by y, and $a = -1/h$. Accordingly, we find that, for $|he| \leq 1$,

$$x = \frac{1}{y} = -\sum_{k=0}^{\infty} \frac{(k-1)^{k-1}(-h)^k}{k!}.$$

On the other hand, putting $a = 1$ in (17.4) yields

$$x = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A_k h^k}{k!}.$$

A comparison of these two series yields the desired result.

Example 2. *Fix* a, $0 < a < e$. *For real* h, *define* $x > 0$ *by the relation* $x^{1/x} = a^{1/a} e^h$. *Then for sufficiently small* $|h|$,

$$\frac{a}{x} = 1 - \sum_{k=1}^{\infty} \frac{A_k (ah)^k}{k!}.$$

Proof. Putting $x = 1/y$, we find that $y^y = (1/a)^{1/a} e^{-h}$. Now use (17.4) with a replaced by $1/a$, h replaced by $-h$, and x replaced by y. The desired equality now easily follows.

F. Howard [1] has shown that the numbers $a(r, k)$ can be expressed in terms of Stirling numbers of the first kind and associated Stirling numbers of the second kind.

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CHAPTER 4

Iterates of the Exponential Function and an Ingenious Formal Technique

The first seven paragraphs of Chapter 4 are concerned with iterated exponential functions and constitute a sequel to a large portion of Chapter 3 wherein the Bell numbers, single-variable Bell polynomials, and related topics are studied. Recall that the Bell numbers $B(n)$, $0 \leq n < \infty$, may be defined by

$$e^{e^x - 1} = \sum_{n=0}^{\infty} \frac{B(n)}{n!} x^n.$$

They were first thoroughly studied in print by Bell [1], [2] approximately 25–30 years after Ramanujan had derived several of their properties in the notebooks. Further iterations of the exponential function appear to have been scarcely studied in the literature. The most extensive study was undertaken by Bell [2] in 1938. Becker and Riordan [1] and Carlitz [1] have established arithmetical properties for these generalizations of Bell numbers. Also, Ginsburg [1] has briefly considered such iterates. For a combinatorial interpretation of numbers generated by iterated exponential functions, see Stanley's article [1, Theorem 6.1].

Sections 9–13 are devoted to a different topic and illustrate one of Ramanujan's favorite techniques. Ramanujan's procedure, however, is strictly formal, and the results that are obtained from it are valid only under severe restrictions. Section 9 contains summation formulas which resemble more complicated formulas found in Chapter 14. In Section 11, we find the formula

$$\int_0^{\infty} x^{n-1} \sum_{k=0}^{\infty} \varphi(k)(-x)^k \, dx = \frac{\pi\varphi(-n)}{\sin(\pi n)}, \qquad n > 0, \tag{0.1}$$

"of which he was especially fond and made continual use." We have quoted

Hardy's book on Ramanujan [20, p. 15], where this formula and others of the same sort in Section 11 are discussed and rigorously proved in Chapter 11.

About one year before Ramanujan's departure from India to England, he obtained a scholarship from the University of Madras [15, pp. xv, xvi]. As stipulated by the scholarship, Ramanujan submitted quarterly reports on his research to the Board of Studies in Mathematics on August 5, 1913, November 7, 1913, and March 9, 1914. The first two reports are concerned entirely with formulas like (0.1) and their applications. Ramanujan gives a detailed proof of a formula (Entry 11) related to (0.1) and even lists hypotheses for which his formula is supposed to be valid. Evidently, early in his stay at Cambridge, Ramanujan discovered, perhaps from Hardy, that his hypotheses were not strong enough. For in [13], [15, pp. 53–58], Ramanujan evidently employs (0.1) to evaluate some integrals and remarks [15, p. 57], "My own proofs of the above results make use of a general formula, the truth of which depends on conditions which I have not yet investigated completely." All of the formulas in Sections 11–13 are established in the quarterly reports, as well as many more.

It is somewhat enigmatic that the integral formulas of Section 11 appear at this location in the second notebook. Since generally the material becomes more difficult and sophisticated in the latter portions of the second notebook, it is reasonable to conjecture that early chapters in the notebooks date from his student days, while later chapters were written in the period just before Ramanujan's departure from India in 1914. The content of the quarterly reports indicates that these integral formulas were discovered at a time later than the other material in Chapter 4. Moreover, Chapter 5 of the first notebook coincides closely with the first eight sections of Chapter 4 of the second notebook, which is a revised, enlarged edition of the first notebook; none of the formulas from Sections 9–15 of Chapter 4 in the second notebook can be found in Chapter 5 of the first notebook. However, Entry 11 can be found on page 182 of the first notebook, which is a "back side" of one of the pages in Chapter 12. Pages 180, 182, and 184 of volume 1 contain 13 applications of Entry 11, only one of which (Entry 13(ii)) is in Chapter 4 of the second notebook.

The primary purpose of Entry 1 is to define the functions that will be studied in the sequel. Let $F_0(x) = x$ and define

$$F_{r+1}(x) = \exp\{F_r(x)\} - 1, \tag{1.1}$$

where r is any integer. Thus, for example, $F_1(x) = e^x - 1$, $F_2(x) = e^{e^x - 1} - 1$, $F_{-1}(x) = \text{Log}(1 + x)$, and $F_{-2}(x) = \text{Log}\{1 + \text{Log}(1 + x)\}$. If r is nonnegative, $F_r(x)$ is entire; if r is negative, $F_r(x)$ is analytic in some neighborhood of the origin. Observe that $F_r(0) = 0$, $-\infty < r < \infty$. Ramanujan tacitly assumes that $F_r(x)$ can be defined for all real values of r. He gives no indication how to do this, but Comtet [1, pp. 144–148] describes how to define arbitrary real iterates of formal power series. In particular, the present situation is specifically discussed on page 147. Thus, Ramanujan writes

$$F_r(x) = \sum_{k=1}^{\infty} \varphi_k(r)x^k = \sum_{k=0}^{\infty} f_k(x)r^k, \tag{1.2}$$

where both series above are to be regarded as formal power series. Comtet [1, p. 147] further shows that $\varphi_k(r)$ is a polynomial in r of degree $k-1$. By making such a substitution in the former series above and collecting coefficients of like powers of r, we easily find that $f_k(x)$ has a zero of order $k+1$ at the origin.

Entry 2. *For each integer r,*

$$F_r'(x) = \{1 + F_r(x)\}F_{r-1}'(x). \tag{2.1}$$

Proof. Differentiate both sides of the relation $F_{r-1}(x) = \operatorname{Log}\{1 + F_r(x)\}$ to achieve the desired equality.

Corollary 1. *If r is a positive integer, then*

$$F_r'(x) = \prod_{k=1}^{r} \{1 + F_k(x)\}.$$

Proof. Apply (2.1) successively a total of r times.

Corollary 2. *Let r be any integer, and suppose that n is a positive integer. Then*

$$n\{\varphi_n(r) - \varphi_n(r-1)\} = \sum_{k=1}^{n-1} (n-k)\varphi_k(r)\varphi_{n-k}(r-1). \tag{2.2}$$

Proof. Using the first equality of (1.2) in (2.1) and equating the coefficients of x^{n-1} on both sides, we readily achieve the desired result.

It might be remarked that Corollary 2 implies that $\varphi_n(r)$ is a polynomial in r of degree $n-1$, a fact previously noted after (1.2). To see this, replace r by j in (2.2) and sum both sides of (2.2) from $j=1$ to $j=r$. Now induct on n to complete the proof.

In preparation for Entry 3, we observe that if f denotes a polynomial, the Euler–Maclaurin formula (I3) yields

$$\sum_{n=0}^{r-1} f(n) = \int_0^r f(x)\,dx + \sum_{n=1}^{\infty} \frac{B_n}{n!}\{f^{(n-1)}(r) - f^{(n-1)}(0)\}, \tag{3.1}$$

where B_n, $1 \le n < \infty$, denotes the nth Bernoulli number.

Entry 3. *For each real number x,*

$$f_1'(x) = x + \sum_{n=1}^{\infty} B_n f_n(x).$$

Proof. Let r be a positive integer. By differentiating (1.1) with respect to x we find that

$$F_r'(x) = e^{F_{r-1}(x)}F_{r-1}'(x) = e^{F_{r-1}(x)+F_{r-2}(x)}F_{r-2}'(x) = \cdots = e^{F_{r-1}(x)+\ldots+F_0(x)},$$

or

$$\operatorname{Log} F_r'(x) = \sum_{k=0}^{r-1} F_k(x). \tag{3.2}$$

Now apply (3.1) to the polynomials $\varphi_k(n)$, $1 \le k < \infty$, to obtain

$$\sum_{n=0}^{r-1} \varphi_k(n) = \int_0^r \varphi_k(u)\,du + \sum_{n=1}^{\infty} \frac{B_n}{n!}\{\varphi_k^{(n-1)}(r) - \varphi_k^{(n-1)}(0)\}. \tag{3.3}$$

Next multiply both sides of (3.3) by x^k and sum on k, $1 \le k < \infty$. Using (1.2) and (3.2), we find that

$$\operatorname{Log} F_r'(x) = \psi(x) + \int_0^r F_u(x)\,du + \sum_{n=1}^{\infty} \frac{B_n}{n!} \left.\frac{d^{n-1}F_u(x)}{du^{n-1}}\right|_{u=r}, \tag{3.4}$$

where $\psi(x)$ depends only upon x and not upon r. Both sides of (3.4) are formal power series in x whose coefficients are polynomials. These polynomial coefficients agree for every positive integer r. Hence, (3.4) is valid for all real numbers r. Now substitute the latter series of (1.2) into (3.4). Equating coefficients of r on both sides and noting that $f_0(x) = x$, we complete the proof of Entry 3.

Corollary. *If x is real, then*

$$\psi(x) = \int_0^x \frac{t - f_1'(t)}{f_1(t)}\,dt.$$

Proof. Setting $r = 0$ in (3.4), we find that

$$\psi(x) = -\sum_{n=1}^{\infty} \frac{B_n}{n} f_{n-1}(x). \tag{3.5}$$

Differentiating (3.5), we deduce that

$$\begin{aligned}
\psi'(x)f_1(x) &= -\sum_{n=1}^{\infty} \frac{B_n}{n} f_{n-1}'(x)f_1(x) \\
&= -\sum_{n=1}^{\infty} B_n f_n(x) \\
&= x - f_1'(x),
\end{aligned} \tag{3.6}$$

by Entry 3. In the penultimate equality, Entry 4 was utilized. Since $f_k(0) = 0$, $0 \le k < \infty$, we find from (3.5) that $\psi(0) = 0$. Hence, from (3.6), we complete the proof.

Entry 4. *For each positive integer n,*

$$nf_n(x) = f_1(x)f'_{n-1}(x). \tag{4.1}$$

Proof. By a general theorem in Comtet's book [1, p. 148],

$$F_{k+r}(x) = F_r\{F_k(x)\}, \tag{4.2}$$

where k and r are arbitrary real numbers. Expanding both sides of (4.2) in power series of r and equating coefficients of r on both sides, we find that $dF_k(x)/dk = f_1\{F_k(x)\}$. Now let $y = F_k(x)$ and $z = F_k(y)$. Then $dy/dk = f_1(y)$ and $dz/dk = f_1(y)\, dz/dy$, or

$$\frac{dF_k(y)}{dk} = f_1(y)\frac{dF_k(y)}{dy}.$$

Using (1.2) above and equating coefficients of k^{n-1} on both sides we deduce (4.1).

We observed after (1.2) that $f_n(x)$ has a $(n+1)$-fold zero at $x = 0$. Hence, we may write

$$f_n(x) = (\tfrac{1}{2}x)^n \sum_{k=1}^{\infty} (-1)^{k-1}\psi_k(n)x^k, \qquad n \geq 0. \tag{4.3}$$

Corollary 1. *If n and r are positive integers, then*

(i)
$$n\psi_r(n) = \sum_{k=1}^{r} (n+k-1)\psi_k(n-1)\psi_{r-k+1}(1)$$

and

(ii)
$$\varphi_n(2r) = \sum_{k=1}^{n} (-1)^{k-1}r^{n-k}\psi_k(n-k).$$

Proof. Substituting (4.3) into (4.1), we arrive at

$$n\sum_{k=1}^{\infty} (-1)^{k-1}\psi_k(n)x^{k-1}$$
$$= \sum_{j=1}^{\infty} (-1)^{j-1}\psi_j(1)x^{j-1} \sum_{k=1}^{\infty} (-1)^{k-1}(n+k-1)\psi_k(n-1)x^{k-1}.$$

Equating coefficients of x^{r-1} on both sides, we complete the proof of (i).

From (1.2) and (4.3),

$$\sum_{n=1}^{\infty} \varphi_n(2r)x^n = \sum_{j=0}^{\infty} f_j(x)(2r)^j = \sum_{j=0}^{\infty} r^j x^j \sum_{k=1}^{\infty} (-1)^{k-1}\psi_k(j)x^k.$$

Equating coefficients of x^n on the extremal sides above, we obtain (ii).

Corollary 2. *We have* $\psi_1(1) = 1$, *and for* $n \geq 2$,

$$(n+1)\psi_n(1) = \tfrac{1}{2}\psi_{n-1}(1) + \sum_{k=1}^{N} B_{2k} 2^{1-2k} \psi_{n-2k}(2k),$$

where N denotes $(n-1)/2$ *or* $(n-2)/2$ *according as n is odd or even.*

Proof. Substitute (4.3) into Entry 3 to obtain

$$\frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n-1}(n+1)\psi_n(1)x^n$$

$$= x - \tfrac{1}{4}x \sum_{n=1}^{\infty} (-1)^{n-1}\psi_n(1)x^n + \sum_{k=1}^{\infty} B_{2k}(\tfrac{1}{2}x)^{2k} \sum_{j=1}^{\infty} (-1)^{j-1}\psi_j(2k)x^j.$$

Equating coefficients of x on both sides, we find that $\psi_1(1) = 1$, and equating coefficients of x^n, $n \geq 2$, on both sides, we deduce the desired formula for $n \geq 2$.

Entry 5. *For each number* $x > -1$, *we have* $f_1(x) = (1+x)f_1\{\text{Log}(1+x)\}$.

Proof. By (4.2),

$$F_{r-1}(x) = F_r\{F_{-1}(x)\} = F_r\{\text{Log}(1+x)\},$$

and so by (1.2) and the fact that $f_0(x) = x$,

$$\exp\{F_{r-1}(x)\} = (1+x)\exp\left\{\sum_{k=1}^{\infty} f_k\{\text{Log}(1+x)\}r^k\right\}. \tag{5.1}$$

On the other hand, by (1.1) and (1.2),

$$\exp\{F_{r-1}(x)\} = 1 + F_r(x) = 1 + \sum_{k=0}^{\infty} f_k(x)r^k. \tag{5.2}$$

Equating the coefficients of r in (5.1) and (5.2), we complete the proof.

Entry 6(i). *For* $n \geq 1$,

$$\frac{1}{n!} = \sum_{k=1}^{n} (-1)^{k-1}2^{k-n}\psi_k(n-k) \tag{6.1}$$

and

$$\frac{1}{n} = \sum_{k=1}^{n} 2^{k-n}\psi_k(n-k). \tag{6.2}$$

Proof. From (1.1) and (1.2),

$$\sum_{n=1}^{\infty} \varphi_n(1)x^n = F_1(x) = e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

Hence, $\varphi_n(1) = 1/n!$. Using Corollary 1(ii) of Entry 4, we deduce (6.1).
Similarly, for $|x| < 1$,

$$\sum_{n=1}^{\infty} \varphi_n(-1)x^n = F_{-1}(x) = \operatorname{Log}(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}.$$

Thus, $\varphi_n(-1) = (-1)^{n-1}/n$. Putting $r = -\frac{1}{2}$ in Corollary 1(ii) of Entry 4, we derive (6.2).

Entry 6(ii). *For* $n \geq 1$,

$$\psi_1(n) = 1, \tag{6.3}$$

$$\psi_2(n) = \frac{n+1}{6} \sum_{k=2}^{n+1} \frac{1}{k}, \tag{6.4}$$

and

$$\psi_3(n) = \frac{(n+1)(n+2)}{72} \left\{ \left(\sum_{k=2}^{n+2} \frac{1}{k} \right)^2 - \sum_{k=2}^{n+2} \frac{1}{k^2} - \frac{1}{n+2} + \frac{1}{2} \right\}. \tag{6.5}$$

Furthermore,

$$\varphi_1(2r) = 1, \qquad \varphi_2(2r) = r, \qquad \varphi_3(2r) = r^2 - \tfrac{1}{6}r,$$

$$\varphi_4(2r) = r^3 - \frac{5}{12}r^2 + \frac{1}{24}r, \qquad \varphi_5(2r) = r^4 - \frac{13}{18}r^3 + \frac{1}{6}r^2 - \frac{1}{90}r,$$

$$\varphi_6(2r) = r^5 - \frac{77}{72}r^4 + \frac{89}{216}r^3 - \frac{91}{1440}r^2 + \frac{11}{4320}r,$$

and

$$\varphi_7(2r) = r^6 - \frac{29}{20}r^5 + \frac{175}{216}r^4 - \frac{149}{720}r^3 + \frac{91}{4320}r^2 - \frac{1}{3360}r.$$

Moreover,

$$f_1(x) = \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{48}x^4 - \frac{1}{180}x^5 + \frac{11}{8640}x^6 - \frac{1}{6720}x^7 + \cdots.$$

Proof. By Corollary 2 of Entry 4, $\psi_1(1) = 1$. Thus, letting $r = 1$ in Corollary 1 of Entry 4, we find that $\psi_1(n) = \psi_1(n-1)$. It then follows that $\psi_1(n) = 1$ for each positive integer n.

By Corollary 2 of Entry 4, $\psi_2(1) = 1/6$. Letting $r = 2$ in Corollary 1 of Entry 4 and using (6.3), we then deduce that

$$\begin{aligned}\psi_2(n) &= \psi_2(1) + \frac{n+1}{n}\psi_2(n-1)\\ &= \frac{1}{6}\left\{1 + \frac{n+1}{n}\right\} + \frac{n+1}{n-1}\psi_2(n-2)\\ &= \frac{1}{6}\left\{1 + \frac{n+1}{n} + \frac{n+1}{n-1} + \cdots + \frac{n+1}{2}\right\}\\ &= \frac{n+1}{6}\sum_{k=2}^{n+1}\frac{1}{k}.\end{aligned}$$

In examining $\psi_3(n)$ it will be convenient to use the function $\chi(n)$ defined by $\psi_3(n) = (n+1)(n+2)\chi(n)$. By Corollary 2 of Entry 4, $\psi_3(1) = 1/24$. Once again by Corollary 1 of Entry 4,

$$\chi(n) = \chi(n-1) + \frac{1}{36(n+2)}\sum_{k=2}^{n}\frac{1}{k} + \frac{1}{24(n+1)(n+2)}.$$

Since $\chi(1) = 1/144$, we deduce that

$$\begin{aligned}\chi(n) &= \frac{1}{144} + \frac{1}{36}\sum_{m=2}^{n}\frac{1}{m+2}\sum_{k=2}^{m}\frac{1}{k} + \frac{1}{24}\sum_{m=2}^{n}\frac{1}{(m+1)(m+2)}\\ &= \frac{1}{144} + \frac{1}{36}\left\{\sum_{m=0}^{n}\frac{1}{m+2}\sum_{k=2}^{m+1}\frac{1}{k} - \frac{1}{6} - \sum_{m=2}^{n}\frac{1}{(m+1)(m+2)}\right\}\\ &\quad + \frac{1}{24}\sum_{m=2}^{n}\frac{1}{(m+1)(m+2)}\\ &= \frac{1}{432} + \frac{1}{72}\left\{\left(\sum_{m=2}^{n+2}\frac{1}{m}\right)^2 - \sum_{m=2}^{n+2}\frac{1}{m^2}\right\} + \frac{1}{72}\left(\frac{1}{3} - \frac{1}{n+2}\right).\end{aligned}$$

Formula (6.5) now readily follows from the equality above.

The functions $\varphi_n(2r)$ may be calculated from formula (ii) of Corollary 1 of Entry 4. Recall that $\psi_k(0) = 0$ if $k \geq 2$. In the calculations of $\psi_1(n)$ and $\psi_2(n)$, employ (6.3) and (6.4), respectively. It is not very convenient to use (6.5) to determine $\psi_3(n)$. Instead, it is easier to employ formula (i) of Corollary 1 of Entry 4 directly. For $\psi_k(n)$, $k \geq 4$, also use the aforementioned formula.

Lastly, the formula for $f_1(x)$ follows from (4.3) and the previously made calculations of $\psi_k(1)$, $1 \leq k \leq 6$. (In the second notebook, Ramanujan inadvertently implies that $f_1(x)$ is a polynomial of degree 7.)

Entry 7. *Let* $x = y(1 - rx)$ *and* $z = 1 - rx$. *Then*

$$F_{2r}(x) = y + \tfrac{1}{6}y^2 \operatorname{Log} z + \tfrac{1}{72}y^3\{\operatorname{Log}^2 z + (1 - \operatorname{Log} z)^2 - z\} + \cdots. \quad (7.1)$$

Proof. By (1.2) and (4.3),

$$F_{2r}(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \left\{ \sum_{k=0}^{\infty} \psi_n(k)(rx)^k \right\} x^n. \tag{7.2}$$

We shall show that the first three terms of (7.2) agree, respectively, with the first three terms given by Ramanujan in (7.1).

By (6.3), the first term in (7.2) is

$$x \sum_{k=0}^{\infty} (rx)^k = \frac{x}{1-rx} = y.$$

By (6.4), the second term in (7.2) is

$$-\frac{x^2}{6} \sum_{k=1}^{\infty} (k+1) \sum_{j=2}^{k+1} \frac{1}{j}(rx)^j. \tag{7.3}$$

On the other hand,

$$\begin{aligned} \tfrac{1}{6}y^2 \operatorname{Log} z &= -\frac{x^2}{6} \sum_{n=1}^{\infty} n(rx)^{n-1} \sum_{j=1}^{\infty} \frac{(rx)^j}{j} \\ &= -\frac{x^2}{6} \sum_{k=1}^{\infty} \sum_{j=1}^{k} \frac{k+1-j}{j}(rx)^k \\ &= -\frac{x^2}{6} \sum_{k=1}^{\infty} \left\{ (k+1) \sum_{j=1}^{k} \frac{1}{j} - k \right\} (rx)^k, \end{aligned}$$

which is readily seen to equal (7.3).

Lastly, a straightforward calculation yields

$$\tfrac{1}{72}y^3\{\operatorname{Log}^2 z + (1-\operatorname{Log} z)^2 - z\} = \frac{x^3}{72(1-rx)^3} \left\{ 3rx + \sum_{n=2}^{\infty} c(n)(rx)^n \right\},$$

where

$$c(n) = \frac{6}{n} + \frac{4}{n} \sum_{j=2}^{n-1} \frac{1}{j}, \qquad n \geq 2.$$

Note that $\psi_3(1) = 1/24$. Thus, to show that the third terms of (7.1) and (7.2) agree, we must show, for $n \geq 2$, that $c(n)$ is equal to the coefficient of $(rx)^n$ in

$$72(1-rx)^3 \sum_{k=0}^{\infty} \psi_3(k)(rx)^k.$$

By (6.5) and a very lengthy calculation, this can be accomplished.

Example 1. *For each real number x,*

$$f_1(x) f_1''(x) = f_1(x) - f_2(x) + \sum_{k=1}^{\infty} (2k+1) B_{2k} f_{2k+1}(x).$$

Proof. Upon differentiating the equality of Entry 3 we find that

$$f_1''(x) = 1 - \tfrac{1}{2}f_1'(x) + \sum_{k=1}^{\infty} B_{2k} f_{2k}'(x).$$

Now multiply both sides by $f_1(x)$ and apply Entry 4 to complete the proof.

We will not record Example 2 which is simply a restatement of Entry 7 when $2r = n$ and $x = 1/n$.

This concludes Ramanujan's theory of iterates of the exponential function. I. N. Baker [1], [2] has made a thorough study of iterates of entire functions with particular attention paid to the exponential function in his second paper. These papers also contain references to work on iterates of arbitrary complex order. But we emphasize that no one but Ramanujan seems to have made a study of the coefficients $\varphi_k(r)$ and $f_k(x)$. A continued development of this theory appears desirable.

Entry 8. *Let* $H_n = \sum_{k=1}^{n} 1/k$. *If* x *is a positive integer, then*

(i) $$\sum_{k=1}^{x} H_k = (x+1)H_x - x,$$

(ii) $$\sum_{k=1}^{x} H_k^2 = (x+1)H_x^2 - (2x+1)H_x + 2x,$$

and

(iii) $$\sum_{k=1}^{x} H_k^3 = (x+1)H_x^3 - \tfrac{3}{2}(2x+1)H_x^2 + 3(2x+1)H_x - 6x + \frac{1}{2}\sum_{k=1}^{x}\frac{1}{k^2}.$$

Proof. Inverting the order of summation, we find that

$$\sum_{k=1}^{n}\sum_{n=1}^{k}\frac{1}{n} = \sum_{n=1}^{x}\frac{1}{n}\sum_{k=n}^{x} 1 = \sum_{n=1}^{x}\frac{x-n+1}{n},$$

from which (i) follows.

By partial summation,

$$\sum_{k=1}^{x} a_k b_k = s_x b_x - \sum_{k=1}^{x-1} s_k(b_{k+1} - b_k), \tag{8.1}$$

where $s_n = \sum_{k=1}^{n} a_k$. Put $a_k = b_k = H_k$ in (8.1) and use (i) to get

$$\begin{aligned}\sum_{k=1}^{x} H_k^2 &= \{(x+1)H_x - x\}H_x - \sum_{k=1}^{x-1}\{(k+1)H_k - k\}\frac{1}{k+1}\\ &= (x+1)H_x^2 - xH_x - \sum_{k=1}^{x-1} H_k + \sum_{k=1}^{x-1}\left(1 - \frac{1}{k+1}\right)\\ &= (x+1)H_x^2 - xH_x - xH_{x-1} + 2x - 2 - \sum_{k=2}^{x}\frac{1}{k}\\ &= (x+1)H_x^2 - (2x+1)H_x + 2x.\end{aligned}$$

Next, in (8.1), put $a_k = H_k^2$ and $b_k = H_k$. Using (i) and (ii), we find that

$$\sum_{k=1}^{x} H_k^3 - (x+1)H_x^3 + (2x+1)H_x^2 - 2xH_x$$
$$= -\sum_{k=1}^{x-1}\left(H_k^2 - \frac{2k+1}{k+1}H_k + \frac{2k}{k+1}\right)$$
$$= -(x+1)H_x^2 + (2x+1)H_x - 2x + H_x^2$$
$$+ \sum_{k=1}^{x}\frac{2k-1}{k}H_{k-1} - 2\sum_{k=1}^{x}\frac{k-1}{k}$$
$$= -xH_x^2 + (2x+1)H_x - 2x + 2(xH_x - x) - 2(x - H_x) - S,$$

where $S = \sum_{k=1}^{x} H_{k-1}/k$. Thus,

$$\sum_{k=1}^{x} H_k^3 = (x+1)H_x^3 - (3x+1)H_x^2 + 3(2x+1)H_x - 6x - S. \tag{8.2}$$

But,

$$H_x^2 = \sum_{k=1}^{x}\frac{1}{k^2} + 2\sum_{1\le n<k\le x}\frac{1}{nk} = \sum_{k=1}^{x}\frac{1}{k^2} + 2S. \tag{8.3}$$

Substituting the value of S obtained from (8.3) into (8.2), we complete the proof of (iii).

Ramanujan begins the ninth section of Chapter 4 with the following statement and solution (abbreviated by sol. below).

> "If two functions of x be equal, then a general theorem can be formed by simply writing $\varphi(n)$ instead of x^n in the original theorem.
>
> sol. Put $x = 1$ and multiply it by $f(0)$, then change x to x, x^2, x^3, x^4 &c and multiply $f'(0)/1!$, $f''(0)/2!$, $f'''(0)/3!$ &c respectively and add up all the results. Then instead of x^n we have $f(x^n)$ for positive as well as negative values of n. Changing $f(x^n)$ to $\varphi(n)$ we can get the result." (Ramanujan actually denotes $n!$ by $\underline{n|}$.)

The formulas obtained by Ramanujan in illustration of his method are valid only under severe restrictions. We shall illustrate his method in detail for just one of the examples that is given. Then we shall give rigorous proofs for each of Ramanujan's results, but with necessarily very restrictive hypotheses.

Consider the relation $\tan^{-1} x + \tan^{-1}(1/x) = \pi/2$. In accordance with Ramanujan's process, we write down the equalities

$$f(0)(\tan^{-1} 1 + \tan^{-1} 1) = \frac{\pi}{2}f(0),$$
$$\frac{f'(0)}{1!}(\tan^{-1} x + \tan^{-1}(1/x)) = \frac{\pi}{2}\frac{f'(0)}{1!},$$
$$\frac{f''(0)}{2!}(\tan^{-1} x^2 + \tan^{-1}(1/x^2)) = \frac{\pi}{2}\frac{f''(0)}{2!},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Replace $\tan^{-1} z$ above by its Maclaurin series in z, where z is any integral power of x. Now add all the equalities above. On the left side we obtain two double series. Invert the order of summation in each double series to find that

$$\sum_{n=0}^{\infty} \frac{(-1)^n\{f(x^{2n+1})+f(x^{-2n-1})\}}{2n+1} = \frac{\pi}{2} f(1).$$

Replace $f(x^k)$ by $\varphi(k)$ to conclude that

$$\sum_{n=0}^{\infty} \frac{(-1)^n\{\varphi(2n+1)+\varphi(-2n-1)\}}{2n+1} = \frac{\pi}{2} \varphi(0).$$

Of course, this formal procedure is fraught with numerous difficulties.

In Examples 1–3, the circular contours are supposed to be oriented in the positive, counterclockwise direction. The choice of circular contours is not strictly necessary; any appropriate sequence of contours may be chosen.

Example 1. *Let $\varphi(z)$ denote an entire function. Suppose that there exists a sequence r_n, $1 \le n < \infty$, of positive numbers tending to ∞ such that*

$$I_n \equiv \frac{1}{2\pi i} \int_{|z|=r_n} \frac{\varphi(z)\, dz}{z \cos(\pi z/2)} = o(1)$$

as r_n tends to ∞. Then

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \frac{(-1)^k \varphi(2k+1)}{2k+1} = \frac{\pi}{2} \varphi(0).$$

Proof. We evaluate I_n by the residue theorem. Letting R_α denote the residue of the integrand at a pole $z = \alpha$, we readily find that $R_0 = \varphi(0)$ and $R_{2k+1} = 2(-1)^{k+1}\varphi(2k+1)/\{\pi(2k+1)\}$ for each integer k. Applying the residue theorem and letting n tend to ∞, we find that

$$0 = \lim_{n\to\infty} I_n = \varphi(0) + \frac{2}{\pi} \lim_{n\to\infty} \sum_{|2k+1|<r_n} \frac{(-1)^{k+1}\varphi(2k+1)}{2k+1},$$

from which the desired result immediately follows.

It is not difficult to see that the hypotheses of Example 1 hold for $\varphi(z) = \exp(iz\theta)$, when $-\pi/2 < \theta < \pi/2$. Example 1 thus yields the familiar expansion

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cos(2n+1)\theta}{2n+1} = \frac{\pi}{4}, \qquad |\theta| < \frac{\pi}{2}.$$

In Example 2, Ramanujan considers $x/(1+x) + (1/x)/(1+1/x) = 1$. He expands $z/(1+z)$ in powers of z, where z is either x or $1/x$, and proceeds in the same fashion as with Example 1 to conclude that

$$\sum_{n=1}^{\infty} (-1)^{n-1}\{\varphi(n) + \varphi(-n)\} = \varphi(0).$$

Example 2. *Let $\varphi(z)$ denote an entire function. Suppose that there is a sequence of positive numbers r_n, $1 \le n < \infty$, tending to ∞ such that*

$$I_n \equiv \frac{1}{2\pi i} \int_{|z|=r_n} \frac{\varphi(z)\,dz}{\sin(\pi z)} = o(1)$$

as n tends to ∞. Then

$$\lim_{N\to\infty} \sum_{\substack{k=-N\\k\neq 0}}^{N} (-1)^{k-1}\varphi(k) = \varphi(0).$$

Proof. The proof follows the same lines as the previous proof. Evaluate I_n by the residue theorem and then let n tend to ∞.

In Example 3 Ramanujan considers the relation $\operatorname{Log}(1+x) - \operatorname{Log}(1+1/x) = \operatorname{Log} x$. Prior to Example 3, Ramanujan remarks that "If $\varphi(n)$ be substituted for x^n, $\varphi'(0)$ must be substituted for $\log_e x$, $\varphi''(0)$ for $(\log_e x)^2$ &c." Subject to the formality of his argument, these substitutions in the equality above are correct. However, if a procedure analogous to that in Examples 1 and 2 is followed, then $f(x^n)$ is replaced by $\varphi(n)$ and $\operatorname{Log} x$ is replaced by 1 to get the desired formula

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\{\varphi(n) - \varphi(-n)\}}{n} = \varphi'(0).$$

Example 3. *Let $\varphi(z)$ denote an entire function. Suppose that there exists a sequence of positive numbers r_n, $1 \le n < \infty$, tending to ∞ such that*

$$\frac{1}{2\pi i} \int_{|z|=r_n} \frac{\varphi(z)\,dz}{z\sin(\pi z)} = o(1)$$

as n tends to ∞. Then

$$\lim_{N\to\infty} \sum_{\substack{k=-N\\k\neq 0}}^{N} \frac{(-1)^{k-1}\varphi(k)}{k} = \varphi'(0).$$

Proof. The proof is similar to that of Example 1.

If in Example 2 we let $\varphi(z) = \sin(z\theta)/z$, where $-\pi < \theta < \pi$, or if in Example 3 we let $\varphi(z) = \exp(iz\theta)$, where $-\pi < \theta < \pi$, we obtain the well-known expansion

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(n\theta)}{n} = \frac{\theta}{2}, \qquad |\theta| < \pi.$$

Example 4. *Let $\varphi(z) = \psi(z)/\Gamma(z+1)$ satisfy the hypotheses of Example 3. Then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\psi(k)}{k!\,k} = \gamma\psi(0) + \psi'(0), \tag{9.1}$$

where γ denotes Euler's constant.

Proof. Observe that $\varphi(-k)=0$ for each positive integer k. Since $\Gamma'(1)=-\gamma$ (e.g., see (10.7) below), we find that $\varphi'(0)=\gamma\psi(0)+\psi'(0)$. Using this information in Example 3, we finish the proof.

Ramanujan, anticipating his theory of divergent series in Chapter 6, calls γ (c in his notation) the constant of $\sum_{k=1}^{\infty} 1/k$. In fact, Ramanujan does not claim equality in (9.1), but writes that the left side of (9.1) is equal to the right side "nearly." Evidently, Ramanujan realizes that his procedure does not always yield an equality, but perhaps an approximation of some sort.

Now suppose that φ is such that the integrals I_n in Examples 1–3 do not tend to 0 as n tends to ∞. The method, however, may still be used to obtain a finite expansion with a remainder term, and possibly an asymptotic expansion may be obtained in certain cases. With something like this in mind, Ramanujan offers a second enigmatic remark prior to Example 3. "If an infinite number of terms vanish it may assume the form $0\times\infty$ and have a definite value. This error in case of a function of x is a function of e^{-x} which rapidly decreases as x increases."

Corollary 1. *As x tends to ∞,*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k!\,k}\sim \operatorname{Log} x+\gamma,$$

where γ denotes Euler's constant.

Proof. By Corollary 2 in Section 2 of Chapter 3,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k!\,k}=e^{-x}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k}\frac{1}{n}\right)\frac{x^k}{k!}. \tag{9.2}$$

By either Example 4 in Section 10 of Chapter 3 or by the third version of Example 2 in Section 10 of this chapter, the right side of (9.2) is asymptotic to $\operatorname{Log} x+\gamma$ as x tends to ∞.

Ramanujan claims that the difference between the left and right sides in Corollary 1 is between e^{-x}/x and $e^{-x}/(1+x)$. This is, indeed, correct, and a proof is deferred until Section 44 of Chapter 12 in the second notebook where Ramanujan returns to the function on the left side of (9.2).

Corollary 2. *Let n be positive. Then as x tends to ∞,*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^{nk}}{k(k!)^n}\sim n\operatorname{Log} x+n\gamma.$$

For $n=1$, Corollary 2 follows from Corollary 1. However, for at least those values of n which are greater than 2, Corollary 2 is false. To show this, first define

$$G_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{nk}}{(k!)^n}.$$

Then

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{nk}}{k(k!)^n} &= n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k!)^n} \int_0^x t^{nk-1}\, dt \\ &= n \int_0^x \frac{1 - G_n(t)}{t}\, dt. \end{aligned} \tag{9.3}$$

By (9.3) and (10.8) below,

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{nk}}{k(k!)^n} - n \operatorname{Log} x - n\gamma \\ &= n\left\{\int_0^x \frac{1 - G_n(t)}{t}\, dt - \int_0^1 \frac{1 - e^{-t}}{t}\, dt + \int_1^{\infty} \frac{e^{-t}}{t}\, dt - \int_1^x \frac{dt}{t}\right\} \\ &= n\left\{\int_0^x \frac{e^{-t} - G_n(t)}{t}\, dt + \int_x^{\infty} \frac{e^{-t}}{t}\, dt\right\} \\ &= n \int_0^x \frac{e^{-t} - G_n(t)}{t}\, dt + o(1), \end{aligned} \tag{9.4}$$

as x tends to ∞. If Corollary 2 is correct, then the integral on the far right side of (9.4) is $o(1)$ as x tends to ∞. But from Olver's text [1, p. 309], for $n \geq 2$,

$$G_n(x) = \frac{2 \exp(xn \cos(\pi/n))}{\sqrt{n}(2\pi x)^{(n-1)/2}} \left\{\sin\left(\frac{\pi}{2n} + xn \sin\frac{\pi}{n}\right) + O\left(\frac{1}{x}\right)\right\},$$

as x tends to ∞. Thus, for $n > 2$ and $\alpha > 0$,

$$\lim_{x \to \infty} \int_{\alpha}^{x} \frac{G_n(t)}{t}\, dt$$

does not exist. In particular, the integral on the far right side of (9.4) does not tend to 0 as x tends to ∞, which disproves Ramanujan's claim.

We are grateful to Emil Grosswald for informing us that our original published proofs of Corollaries 1 and 2 are incorrect.

Ramanujan expresses Corollary 2 in the words, "If x becomes greater and greater

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{nk}}{k(k!)^n} = n \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k!\, k}."$$

In more precise terminology, Ramanujan is saying that the two series above are asymptotic to the same function as x tends to ∞. If Corollary 2 were correct, this would be the case, but we have just seen that, in general, Corollary 2 is false.

If n is a nonnegative integer, the next entry is trivial with no hypotheses needed for φ. Ramanujan "proves" Entry 10 by the technique described in the ninth section. The starting point of his argument is the identity $(1+x)^n = x^n(1+1/x)^n$. We employ the same notation and conventions for circular contours as in Section 9.

Entry 10. *Let n be complex and let φ be an entire function. Suppose that there exists a sequence r_j, $1 \le j < \infty$, of positive numbers tending to ∞ such that*

$$I_j \equiv \frac{1}{2\pi i}\int_{|z|=r_j} \frac{\pi\varphi(z)\cot(\pi z)\cot\{\pi(z-n)\}}{\Gamma(z+1)\Gamma(n-z+1)}\,dz = o(1)$$

as r_j tends to ∞. Then

$$\sum_{k=0}^{\infty}\binom{n}{k}\varphi(k) = \sum_{k=0}^{\infty}\binom{n}{k}\varphi(n-k),$$

provided each series converges.

Proof. Since the theorem is trivial when n is a nonnegative integer, we assume the contrary in the remainder of the proof.

The integrand of I_j has simple poles at $z = k$, where k is a nonnegative integer, and at $z = n + r$, where r is a nonpositive integer. Easy calculations yield

$$R_k = -\frac{\varphi(k)\cot(\pi n)}{\Gamma(k+1)\Gamma(n-k+1)}, \qquad k \ge 0,$$

and

$$R_{n+r} = \frac{\varphi(n+r)\cot\{\pi(n+r)\}}{\Gamma(n+r+1)\Gamma(-r+1)} = \frac{\varphi(n-k)\cot(\pi n)}{\Gamma(n-k+1)\Gamma(k+1)}, \qquad k = -r \ge 0.$$

Applying the residue theorem, letting j tend to ∞, and invoking the hypothesis on I_j, we find that

$$\sum_{k=0}^{\infty}\left\{-\frac{\varphi(k)\cot(\pi n)}{\Gamma(k+1)\Gamma(n-k+1)} + \frac{\varphi(n-k)\cot(\pi n)}{\Gamma(k+1)\Gamma(n-k+1)}\right\} = 0,$$

from which the theorem's conclusion follows.

Corollary. *Let the hypotheses of Entry 10 be satisfied with n replaced by r and $\varphi(z)$ replaced by $x^{r-z}\varphi(z)$, where $x > 0$. Let*

$$\sum_{k=0}^{\infty}\binom{r}{k}x^{r-k}\varphi(k) \tag{10.1}$$

represent a function f in a neighborhood of ∞. Then $f(0) = \varphi(r)$.

Proof. Replacing $\varphi(z)$ by $x^{r-z}\varphi(z)$ and letting $n = r$ in Entry 10, we find that

$$f(x) = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k}\varphi(k) = \sum_{k=0}^{\infty} \binom{r}{k} x^{k}\varphi(r-k).$$

Setting $x = 0$ on the far right side above, we complete the proof.

Example 1. *Let the hypotheses of the Corollary above be satisfied with* $r = -1$ *and* $\varphi(z)$ *replaced by* $\varphi(z+1)$. *Then* $f(0) = \varphi(0)$.

Proof. The conclusion above is a direct consequence of the Corollary.

It is clear that the conditions in the Corollary and Example 1 are rarely satisfied. In many instances (10.1) does not converge but may be an asymptotic expansion for a function f. The Corollary and Example 1 then become ambiguous because many functions have the same asymptotic series, when that series diverges. Ramanujan calls f "the generating function of the series" (10.1) (p. 43). In the examples which follow, Ramanujan gives applications of his theory above. However, as we have intimated, his formulations are usually ambiguous and his proofs generally are not rigorous. In each case, we shall offer a precise statement of Ramanujan's example and give a rigorous proof.

Example. *The function represented by*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)x^{2k+1}}, \qquad |x| > 1, \tag{10.2}$$

has the value $\pi/2$ *as* x *tends to* 0, $x > 0$.

Proof. The series (10.2) is the Maclaurin series for $\tan^{-1}(1/x)$, $|x| > 1$, and so there is no ambiguity in this example. Since $\lim_{x\to 0+} \tan^{-1}(1/x) = \pi/2$, the result is established.

Ramanujan made his deduction by setting $\varphi(z) = \sin(\pi z/2)/z$ in Example 1.

Example 2 (First Version). *As* $x > 0$ *tends to* ∞,

$$f(x) \equiv \int_0^{\infty} \frac{e^{-t}}{x+t}\, dt \sim \sum_{k=0}^{\infty} \frac{(-1)^k k!}{x^{k+1}}, \tag{10.3}$$

and $\lim_{x\to 0+} f(x) = \infty$.

Proof. The asymptotic series in (10.3) is readily obtained by successive integrations by parts. This result was actually first obtained by Euler, and a very thorough discussion of it can be found in Hardy's book [15, pp. 26–27].

Integrating by parts (in a manner different from that indicated above), we get

$$f(x) = \frac{1}{x} + \int_0^\infty e^{-t} \operatorname{Log}(x+t)\,dt,$$

from which the latter conclusion of Example 2 follows. Ramanujan deduces this result by setting $\varphi(z) = \Gamma(z)$ in Example 1.

Example 2 (Second Version). *With f as above and $x > 0$,*

$$f(x) = \frac{1}{x+1} \, - \, \frac{1^2}{x+3} \, - \, \frac{2^2}{x+5} \, - \, \frac{3^2}{x+7} \, - \, \cdots,$$

and $\lim_{x \to 0+} f(x) = \infty$.

Proof. This continued fraction expansion for f is fairly well known and can be found in Wall's book [1, p. 356].

To show that $f(0+) = \infty$, Ramanujan evidently reasoned as follows. Set $x = 0$ in the nth convergent of the continued fraction expansion above. Now there exists a standard algorithm, which can be found in Wall's treatise [1, pp. 17–18], for converting continued fractions into series. Accordingly, we find that

$$\frac{1}{1} \, - \, \frac{1^2}{3} \, - \, \frac{2^2}{5} \, - \, \cdots \, - \, \frac{(n-1)^2}{2n-1} = \sum_{k=1}^{n} \frac{1}{k}.$$

Letting n tend to ∞, we find that $f(0+) = \infty$.

Example 2 (Third Version). *Let*

$$f(x) = e^x \int_x^\infty \frac{e^{-t}}{t}\,dt$$

for $x > 0$. Then

$$\sum_{k=1}^{\infty} \frac{\psi(k)x^k}{k!} - e^x(\gamma + \operatorname{Log} x) = f(x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k k!}{x^{k+1}}, \tag{10.4}$$

as x tends to ∞, where $\psi(k) = \sum_{n \le k} 1/n$ and γ denotes Euler's constant. Also, $f(0+) = \infty$.

Proof. The value for $f(0+)$ is immediate from the proposed formula (10.4).

From Corollary 2 of Entry 2 in Chapter 3,

$$\sum_{k=1}^{\infty} \frac{\psi(k)x^k}{k!} = e^x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k!\,k}, \qquad |x| < \infty. \tag{10.5}$$

Now,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k!\,k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^x t^{k-1}\,dt = \int_0^x \frac{1-e^{-t}}{t}\,dt. \tag{10.6}$$

Secondly, from the well-known evaluation (e.g., see Gradshteyn and Ryzhik's book [1], p. 946),

$$\gamma = -\Gamma'(1) = -\int_0^{\infty} e^{-t}\operatorname{Log} t\,dt, \tag{10.7}$$

it is easily shown, by an integration by parts, that

$$\gamma = \int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^{\infty} \frac{e^{-t}}{t}\,dt. \tag{10.8}$$

Using (10.5), (10.6), and (10.8), we find that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\psi(k)x^k}{k!} - e^x(\gamma + \operatorname{Log} x)\\
&\quad = e^x\left\{\int_0^x \frac{1-e^{-t}}{t}\,dt - \int_0^1 \frac{1-e^{-t}}{t}\,dt + \int_1^{\infty} \frac{e^{-t}}{t}\,dt - \int_1^x \frac{dt}{t}\right\}\\
&\quad = e^x \int_x^{\infty} \frac{e^{-t}}{t}\,dt.
\end{aligned}$$

Successively integrating the integral on the far right side above by parts, we readily obtain the desired asymptotic series.

Example 3. *For* Re $n > -1$ *and* $x \geq 0$,

$$f(x) \equiv \int_0^{\infty} e^{-t}(x+t)^n\,dt \sim \sum_{k=0}^{\infty} \frac{\Gamma(n+1)x^{n-k}}{\Gamma(n-k+1)}$$

as x *tends to* ∞. *Furthermore,* $f(0) = \Gamma(n+1)$.

Proof. The proposed value for $f(0)$ is immediate. Ramanujan deduced this value by letting $\varphi(z) = \Gamma(z+1)$ and $r = n$ in the Corollary.

Applying Taylor's theorem, we find that for each nonnegative integer N and for some value t_0, $0 < t_0 = t_0(t, N) < x$,

$$\begin{aligned}
f(x) &= x^n \int_0^{\infty} e^{-t}\left\{\sum_{k=0}^{N} \binom{n}{k}\left(\frac{t}{x}\right)^k + \binom{n}{N+1}\left(\frac{t}{x}\right)^{N+1}\left(1+\frac{t_0}{x}\right)^{n-N-1}\right\} dt\\
&= \sum_{k=0}^{N} \frac{\Gamma(n+1)x^{n-k}}{\Gamma(n-k+1)} + O\left(|x|^{\operatorname{Re} n-N-1} \int_0^{\infty} e^{-t}t^{N+1}\left(1+\frac{t_0}{x}\right)^{\operatorname{Re} n-N-1} dt\right)\\
&= \sum_{k=0}^{N} \frac{\Gamma(n+1)x^{n-k}}{\Gamma(n-k+1)} + O(|x|^{\operatorname{Re} n-N-1})
\end{aligned}$$

as x tends to ∞. In the penultimate equality we estimated the integral as follows. If $\operatorname{Re} n - N - 1 \leq 0$, replace $1 + t_0/x$ by 1. If $\operatorname{Re} n - N - 1 > 0$, divide the interval of integration into $[0, 1]$ and $[1, \infty)$. The integral over $[0, 1]$ is obviously bounded as a function of x. Use the inequality $1 + t_0/x < 2t$, for x, $t \geq 1$, to show that the second integral is also bounded. The desired asymptotic series now follows.

Example 4. *For $x \geq 0$, let*

$$f(x) = \int_x^\infty \frac{\sin(t-x)}{t}\,dt.$$

Then

$$\frac{\pi}{2}\cos x + (\gamma + \operatorname{Log} x)\sin x - \sum_{k=0}^\infty \frac{(-1)^k \psi(2k+1) x^{2k+1}}{(2k+1)!}$$
$$= f(x) \sim \sum_{k=0}^\infty \frac{(-1)^k (2k)!}{x^{2k+1}},$$

as x tends to ∞, where ψ is defined in the third version of Example 2. Furthermore, $f(0) = \pi/2$.

Proof. The proposed value for $f(0)$ follows either from the definition of f or the proposed equality for f. Ramanujan evidently put $\varphi(z) = \Gamma(z)\sin(\pi z/2)$ in Example 1 to achieve the value of $f(0)$.

To establish the desired equality for f, first replace x by ix in (10.5) and equate imaginary parts to obtain

$$\sum_{k=0}^\infty \frac{(-1)^k \psi(2k+1) x^{2k+1}}{(2k+1)!}$$
$$= \cos x \sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{(2k+1)!\,(2k+1)} - \sin x \sum_{k=1}^\infty \frac{(-1)^k x^{2k}}{(2k)!\,(2k)}. \tag{10.9}$$

Now,

$$\sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{(2k+1)!\,(2k+1)} = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!}\int_0^x t^{2k}\,dt = \int_0^x \frac{\sin t}{t}\,dt. \tag{10.10}$$

Similarly,

$$\sum_{k=1}^\infty \frac{(-1)^k x^{2k}}{(2k)!\,(2k)} = \int_0^x \frac{\cos t - 1}{t}\,dt. \tag{10.11}$$

Thus, employing (10.9)–(10.11), we find that

$$\frac{\pi}{2}\cos x+(\gamma+\operatorname{Log} x)\sin x-\sum_{k=0}^{\infty}\frac{(-1)^k\psi(2k+1)x^{2k+1}}{(2k+1)!}$$

$$=\frac{\pi}{2}\cos x+(\gamma+\operatorname{Log} x)\sin x-\cos x\int_0^x\frac{\sin t}{t}\,dt+\sin x\int_0^x\frac{\cos t-1}{t}\,dt$$

$$=\cos x\int_x^{\infty}\frac{\sin t}{t}\,dt+\sin x\left\{\gamma+\operatorname{Log} x+\int_0^x\frac{\cos t-1}{t}\,dt\right\}$$

$$=\cos x\int_x^{\infty}\frac{\sin t}{t}\,dt-\sin x\int_x^{\infty}\frac{\cos t}{t}\,dt$$

$$=\int_x^{\infty}\frac{\sin(t-x)}{t}\,dt.$$

The penultimate equality was achieved by using a familiar formula for the cosine integral. (See Gradshteyn and Ryzhik's book [1], p. 928.)

Finally, the asymptotic series representation for f is easily established by repeated integrations by parts.

After the examples, Ramanujan points out the "advantage" of his method by remarking, "Thus we are able to find exact values when $x=0$, though the generating functions may be too difficult to find."

Entry 11. *If* $n>0$, *then*

$$\int_0^{\infty}x^{n-1}\sum_{k=0}^{\infty}\frac{\varphi(k)(-x)^k}{k!}\,dx=\Gamma(n)\varphi(-n).$$

Corollary 1. *If* $n>0$, *then*

$$\int_0^{\infty}x^{n-1}\sum_{k=0}^{\infty}\varphi(k)(-x)^k\,dx=\frac{\pi\varphi(-n)}{\sin(\pi n)}.$$

Corollary 2. *If* $n>0$, *then*

$$\int_0^{\infty}x^{n-1}\sum_{k=0}^{\infty}\frac{\varphi(2k)(-x^2)^k}{(2k)!}\,dx=\Gamma(n)\varphi(-n)\cos\left(\frac{\pi n}{2}\right).$$

Corollary 3. *If* n *is real, then*

$$\int_0^{\infty}\sum_{k=0}^{\infty}\frac{\varphi(k)(-x)^k}{k!}\cos(nx)\,dx=\sum_{k=0}^{\infty}\varphi(-2k-1)(-n^2)^k.$$

Corollary 4. *If* n *is real, then*

$$\int_0^{\infty}\sum_{k=0}^{\infty}\varphi(2k)(-x^2)^k\cos(nx)\,dx=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\varphi(-k-1)(-n)^k}{k!}.$$

Ramanujan's deduction of the five equalities above is strictly formal. First, he deduces a familiar integral representation for $\Gamma(n+1)$ by employing successive integrations by parts and Example 3 of Entry 10, not the corollary of Entry 10 as was claimed. To establish Entry 11, Ramanujan considers

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \int_0^{\infty} e^{-r^k x} x^{n-1}\, dx = \sum_{k=0}^{\infty} \frac{\Gamma(n) f^{(k)}(0)}{r^{kn} k!},$$

where $r, n > 0$. Invert the order of summation and integration. Then expand $\exp(-r^k x)$ in its Maclaurin series and invert the order of summation to obtain

$$\int_0^{\infty} x^{n-1} \sum_{k=0}^{\infty} \frac{f(r^k)(-x)^k}{k!}\, dx = \Gamma(n) f(r^{-n}).$$

Replacing $f(r^j)$ by $\varphi(j)$, $j = -n, k$, $0 \le k < \infty$, Ramanujan completes his derivation.

Corollary 1 is obtained from Entry 11 by replacing $\varphi(x)$ by $\Gamma(x+1)\varphi(x)$. To establish Corollary 2, replace $\varphi(x)$ by $\varphi(x)\cos(\pi x/2)$ in Entry 11. To deduce Corollary 3, expand $\cos(nx)$ in its Maclaurin series and invert the order of integration and summation for this series. Appealing to Entry 11, we complete the argument. Corollary 4, which also appears as Corollary (vii) in Ramanujan's second quarterly report, is more difficult to prove. We defer a proof and full discussion of Corollary 4 until our examination of the quarterly reports.

In the introductory chapter of his book [20, p. 15] on the work of Ramanujan, Hardy quotes Corollary 1 and remarks that this "particularly interesting formula" was one of Ramanujan's favorite formulas. Later, in Chapter 11, Hardy discusses Entry 11, Corollary 1, and Corollary 3 very thoroughly and establishes their validity for certain classes of the functions φ. In regard to Ramanujan's possible proofs of these formulas, Hardy remarks, "but he had not 'really' proved any of the formulas which I have quoted. It was impossible that he should have done so because the 'natural' conditions involve ideas of which he knew nothing in 1914, and which he had hardly absorbed before his death." Complete proofs of Ramanujan's formulas may also be found in Hardy's paper [13], [21, pp. 280–289]. For some applications of Corollary 1 to the Riemann zeta-function, see H. M. Edwards' book [1, pp. 218–225]. Another formal application of Corollary 1 has been given by Hill, Laird, and Cerone [1].

Corollary 5. *If* $m, n > -1$, *then*

$$\int_0^1 x^m (1-x)^n\, dx = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}.$$

Of course, Corollary 5 is an extremely well-known formula for the beta-function $B(m+1,\ n+1)$. This example is also given in Ramanujan's first quarterly report, and we defer Ramanujan's proof until then.

Entry 12. *If*

$$\psi(n) = \int_0^\infty \varphi(x)\cos(nx)\,dx,$$

then

$$\text{(i)} \qquad \int_0^\infty \psi(x)\cos(nx)\,dx = \frac{\pi}{2}\varphi(n)$$

and

$$\text{(ii)} \qquad \int_0^\infty \psi^2(x)\,dx = \frac{\pi}{2}\int_0^\infty \varphi^2(x)\,dx.$$

The first part of Entry 12 gives the inversion formula for Fourier cosine transforms. Sufficient conditions for the validity of (i) and (ii) can be found in Chapters 3 and 2, respectively, of Titchmarsh's book [2]. We defer Ramanujan's "proof" until our examination of his quarterly reports wherein several applications of Entry 12 are given.

Entry 13(i). *If* $m, n > -1$, *then*

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2}+1\right)}.$$

The result above is well known and is an easy consequence of Corollary 5 of Entry 11. The formula below is also well known. One of the simplest proofs is found in Whittaker and Watson's treatise [1, exercise 39, p. 263].

Entry 13(ii). *If* $m > -1$ *and* n *is any complex number, then*

$$\int_0^{\pi/2} \cos^m x \cos nx\,dx = \frac{\pi\Gamma(m+1)}{2^{m+1}\Gamma\left(\frac{m+n}{2}+1\right)\Gamma\left(\frac{m-n}{2}+1\right)}.$$

Entry 14. *If* x *is any complex number, then*

$$\prod_{n=1}^\infty \left(1+\frac{x^6}{n^6}\right) = \frac{\sinh(2\pi x) - 2\sinh(\pi x)\cos(\pi x\sqrt{3})}{4\pi^3 x^3}.$$

Proof. Letting $\omega = \exp(2\pi i/3)$ and employing the familiar product representation for $\sinh x$, we find that

$$\prod_{n=1}^{\infty}\left(1+\frac{x^6}{n^6}\right) = \frac{\sinh(\pi x)\sinh(\pi x\omega)\sinh(\pi x\omega^2)}{\pi^3 x^3}$$

$$= \frac{\sinh(\pi x)}{\pi^3 x^3}\left\{\sinh^2\left(\frac{\pi x}{2}\right)\cos^2\left(\frac{\pi x\sqrt{3}}{2}\right) + \cosh^2\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x\sqrt{3}}{2}\right)\right\}$$

$$= \frac{\sinh(\pi x)}{4\pi^3 x^3}\{(\cosh(\pi x)-1)(\cos(\pi x\sqrt{3})+1)$$

$$+ (\cosh(\pi x)+1)(1-\cos(\pi x\sqrt{3}))\},$$

from which the desired evaluation follows.

Entry 15. *For each positive integer k, let* $G_k = \sum_{0\le 2n+1\le k} 1/(2n+1)$. *Then for all complex x,*

$$e^x \sum_{k=1}^{\infty} \frac{(-2)^{k-1}x^k}{k!\,k} = \sum_{k=1}^{\infty} \frac{G_k x^k}{k!}. \tag{15.1}$$

Proof. We have

$$I \equiv e^x \int_0^1 \frac{1-e^{-2xz}}{2z}\,dz = e^x \int_0^1 \sum_{k=1}^{\infty} \frac{x^k(-2z)^{k-1}}{k!}\,dz = e^x \sum_{k=1}^{\infty} \frac{(-2)^{k-1}x^k}{k!\,k}. \tag{15.2}$$

On the other hand,

$$I = \int_0^1 \frac{e^x - e^{x(1-2z)}}{2z}\,dz = \sum_{k=1}^{\infty} \frac{x^k}{k!}\int_0^1 \frac{1-(1-2z)^k}{2z}\,dz$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^k}{k!}\int_{-1}^1 \frac{1-t^k}{1-t}\,dt = \sum_{k=1}^{\infty} \frac{x^k}{k!}\int_0^1 \sum_{0\le j\le (k-1)/2} t^{2j}\,dt = \sum_{k=1}^{\infty} \frac{G_k x^k}{k!}. \tag{15.3}$$

Combining (15.2) and (15.3), we deduce (15.1).

Note that by equating coefficients of x^n in (15.1), we find that

$$\sum_{k=1}^{n} \binom{n}{k}\frac{(-2)^{k-1}}{k} = G_n, \qquad n \ge 1.$$

CHAPTER 5

Eulerian Polynomials and Numbers, Bernoulli Numbers, and the Riemann Zeta-Function

Chapter 5 contains more number theory than any of the remaining 20 chapters. Of the 94 formulas or statements of theorems in Chapter 5, the great majority pertain to Bernoulli numbers, Euler numbers, Eulerian polynomials and numbers, and the Riemann zeta-function. As is to be expected, most of these results are not new. The geneses of Ramanujan's first published paper [4] (on Bernoulli numbers) and fourth published paper [7] (on sums connected with the Riemann zeta-function) are found in Chapter 5. Most of Ramanujan's discoveries about Bernoulli numbers that are recorded here may be found in standard texts, such as those by Bromwich [1], Nielsen [5], Nörlund [2], and Uspensky and Heaslet [1], for example.

The notations for the Eulerian polynomials and numbers are not particularly standard, and so we shall employ Ramanujan's notations. Define the Eulerian polynomials $\psi_n(p)$, $0 \leq n < \infty$, $p \neq -1$, by

$$\frac{1}{e^x + p} = \sum_{n=0}^{\infty} \frac{(-1)^n \psi_n(p) x^n}{n!\,(p+1)^{n+1}}, \qquad |x| < |\mathrm{Log}(-p)|. \tag{0.1}$$

It will be shown in the sequel that, indeed, $\psi_n(p)$ is a polynomial in p of degree $n-1$. In the notation of Carlitz's paper [3], which is perhaps the most extensive source of information about Eulerian polynomials and numbers, $R_n[-p] = \psi_n(p)$. The Eulerian numbers A_{nk}, $1 \leq k \leq n$, are generally defined by

$$\psi_n(p) = \sum_{k=1}^{n} A_{nk}(-p)^{k-1}.$$

In Ramanujan's notation $A_{nk} = F_k(n)$; see (6.1) below. The Eulerian polynomials and numbers were first introduced by Euler [1] in 1755. Carlitz [3], [6], [7], and jointly with his colleagues Kurtz, Scoville, and Stackelberg [1], Riordan [1], and Scoville [3], [4], has written extensively about Eulerian polynomials and numbers and certain generalizations thereof. See also a paper of Frobenius [1], [2, pp. 809–847], for much historical information, and Riordan's book [1], which contains combinatorial applications. In particular, A_{nk} is equal to the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k rises including a conventional one at the left. Some of Ramanujan's theorems on Eulerian polynomials and numbers appear to be new. Also, since most results in this area are not well known and the proofs are very short, we shall give most proofs.

In Entries 1(i), 1(ii), and 3 it is assumed that, for $|h|$ sufficiently small, f can be expanded in the form

$$f(x) = \sum_{n=0}^{\infty} a_n h^n \varphi^{(n)}(x), \tag{1.1}$$

where a_n, $0 \leq n < \infty$, is independent of φ.

Entry 1(i). *Let $f(x+h) - f(x) = h\varphi'(x)$. Then, in the notation of (1.1), $a_n = B_n/n!$, $0 \leq n < \infty$, where B_n denotes the* n*th Bernoulli number.*

Proof. Since a_n, $0 \leq n < \infty$, is independent of φ, let $\varphi(x) = e^x$. Then, by (1.1),

$$he^x = f(x+h) - f(x) = e^x(e^h - 1) \sum_{n=0}^{\infty} a_n h^n,$$

i.e.,

$$\sum_{n=0}^{\infty} a_n h^n = \frac{h}{e^h - 1}. \tag{1.2}$$

Comparing (1.2) with (I1), we complete the proof.

Entry 1(ii). *Let $f(x+h) + f(x) = h\varphi'(x)$. Then $a_n = (1 - 2^n)B_n/n!$, $0 \leq n < \infty$.*

Proof. Again, without loss of generality, we take $\varphi(x) = e^x$. Then, from (1.1),

$$he^x = f(x+h) + f(x) = e^x(e^h + 1) \sum_{n=0}^{\infty} a_n h^n.$$

Thus, from (I1),

$$\sum_{n=0}^{\infty} a_n h^n = \frac{h}{e^h + 1} = \frac{h}{e^h - 1} - \frac{2h}{e^{2h} - 1} = \sum_{n=0}^{\infty} \frac{(1 - 2^n)B_n h^n}{n!}, \tag{1.3}$$

and the desired result follows.

Suppose that f is a solution of either the difference equation of Entry 1(i) or of Entry 1(ii). Then, in general, the series on the right side of (1.1) diverges. However, Nörlund [2, pp. 58–60] has shown that under suitable conditions the series on the right side of (1.1) represents the function asymptotically as h tends to 0.

Let φ be any function defined on $(-\infty, \infty)$. In anticipation of Entries 2(i) and 2(ii), define

$$F_{2n+1}(x) = \varphi(x) + \sum_{k=1}^{n} (-1)^k \frac{(n!)^2}{(n+k)!\,(n-k)!} \{\varphi(x+k) + \varphi(x-k)\}, \quad (2.1)$$

where n is a nonnegative integer.

Entry 2(i). *Let $f(x+1) - f(x-1) = 2\varphi'(x)$, where φ is a polynomial. Then there exists a polynomial solution f of the form*

$$f(x) = \sum_{n=0}^{\infty} \frac{F_{2n+1}(x)}{2n+1}, \quad (2.2)$$

where F_{2n+1} is defined by (2.1).

Entry 2(ii). *Let $f(x+1) + f(x-1) = 2\varphi(x)$, where φ is a polynomial. Then there exists a polynomial solution f of the form*

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} \binom{2n}{n} F_{2n+1}(x), \quad (2.3)$$

where F_{2n+1} is defined by (2.1).

Before embarking upon the proofs of these theorems we shall make several comments.

First, the series on the right sides of (2.2) and (2.3) are, in fact, finite series. This will be made evident in the proofs of (2.2) and (2.3). In some cases, the restrictions that f and φ be polynomials may be lifted. However, in general, if f is a solution of the difference equation in Entry 2(i) or Entry 2(ii), the series on the right side of (2.2) or (2.3), respectively, does not converge.

Ramanujan actually considers the seemingly more general difference equations $f(x+h) - f(x-h) = 2h\varphi'(x)$ and $f(x+h) + f(x-h) = 2\varphi(x)$ in Entries 2(i) and 2(ii), respectively. However, it is no loss of generality to assume that $h = 1$. For suppose, for example, that Entry 2(i) has been established. Put $y = xh$, $f(x) = g(y)$, and $\varphi(x) = \psi(y)$. Then the difference equation of Entry 2(i) becomes $g(y+h) - g(y-h) = 2h\psi'(y)$.

We are very grateful to Doron Zeilberger for suggesting the following method of proof for Entries 2(i) and 2(ii). Since the proofs use operator calculus, we need to define a couple of operators. As customary, let D denote the differential operator and define E by $Ef(x) = f(x+1)$. Note that $E = e^D$.

Proof of Entry 2(i). We shall first derive another formulation of $F_{2n+1}(x)$. From (2.1),

$$\begin{aligned} F_{2n+1}(x) &= \sum_{k=-n}^{n} \frac{(-1)^k (n!)^2 E^k \varphi}{(n+k)!\,(n-k)!} \\ &= \frac{(n!)^2(-E)^n}{(2n)!} \sum_{j=0}^{2n} \binom{2n}{j} (-E^{-1})^{2n-j} \varphi \\ &= \frac{(n!)^2(-1)^n}{(2n)!} e^{nD}(1-e^{-D})^{2n}\varphi \\ &= \frac{(-1)^n (n!)^2}{(2n)!} \left(2 \sinh\left(\frac{D}{2}\right)\right)^{2n} \varphi. \end{aligned} \tag{2.4}$$

In operator notation, the given difference equation is $(E-E^{-1})f = 2D\varphi$, or $f = 2D\varphi/(e^D - e^{-D})$. Hence, in operator notation, the proposed identity may be written as

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(2n+1)!} \left(2 \sinh\left(\frac{D}{2}\right)\right)^{2n+1} \varphi = \frac{2D\varphi}{e^{D/2} + e^{-D/2}}. \tag{2.5}$$

Set $y = \sinh(D/2) = (\sqrt{E} - 1/\sqrt{E})/2$. (Note that $(\sqrt{E} - 1/\sqrt{E})^n \varphi = 0$ if n exceeds the degree of φ, and so the series in (2.5), indeed, does terminate.) A short calculation shows that $\sqrt{y^2+1} = (e^{D/2} + e^{-D/2})/2$. Hence, it suffices to prove that

$$\sqrt{y^2+1} \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2 (2y)^{2n+1} \varphi}{(2n+1)!} = 2(\sinh^{-1} y)\varphi. \tag{2.6}$$

Obviously, (2.6) is valid for $y = 0$. Thus, it suffices to show that the derivatives of both sides of (2.6) are equal. After taking derivatives in (2.6) and multiplying both sides by $\frac{1}{2}\sqrt{y^2+1}$, we find that it is sufficient to prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2 2^{2n} y^{2n+2} \varphi}{(2n+1)!} + (y^2+1) \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2 (2y)^{2n} \varphi}{(2n)!} = \varphi. \tag{2.7}$$

Combining the series together on the left side of (2.7), we easily obtain φ after a short calculation. This completes the proof.

Proof of Entry 2(ii). Using (2.4), we find that the left side of (2.3) may be written as

$$\begin{aligned} \sum_{n=0}^{\infty} \left(-2 \sinh^2\left(\frac{D}{2}\right)\right)^n \varphi &= \frac{1}{1 + 2\sinh^2(D/2)} \varphi \\ &= \frac{1}{\cosh D} \varphi. \end{aligned}$$

Since the given difference equation in operator notation is $(E + E^{-1})f = 2\varphi$ or $f = \varphi/\cosh D$, the desired result follows.

Entry 3. *Let $f(x+h)+pf(x)=\varphi(x)$, where $p\neq -1$. Then, in the notation of* (1.1),

$$a_n=\frac{(-1)^n\psi_n(p)}{n!\,(p+1)^{n+1}},\qquad 0\le n<\infty,$$

where $\psi_n(p)$ is defined by (0.1).

Proof. As before, since a_n, $0\le n<\infty$, is independent of φ, we shall let $\varphi(x)=e^x$. Using (1.1) and proceeding in the same fashion as in the proof of Entry 1(i), we find that

$$\sum_{n=0}^{\infty} a_n h^n=\frac{1}{e^h+p}. \tag{3.1}$$

Comparing (3.1) with (0.1), we deduce the desired result.

Appell [1], Brodén [1], [2], and Picard [1] have studied periodic solutions of $f(x+h)+pf(x)=\varphi(x)$. For a discussion of solutions to many types of difference equations and for numerous references, see Nörlund's article in the *Encyklopädie* [1].

Entry 4. *For $|p|<1$ and $n\ge 0$,*

$$(p+1)^{-n-1}\psi_n(p)=\sum_{k=0}^{\infty}(k+1)^n(-p)^k. \tag{4.1}$$

Proof. For $|e^{-x}p|<1$,

$$\frac{1}{e^x+p}=\sum_{k=0}^{\infty}e^{-(k+1)x}(-p)^k=\sum_{k=0}^{\infty}(-p)^k\sum_{n=0}^{\infty}\frac{(k+1)^n(-x)^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\sum_{k=0}^{\infty}(k+1)^n(-p)^k. \tag{4.2}$$

If we now equate coefficients of x^n, $n\ge 0$, in (0.1) and (4.2), we deduce (4.1). However, this procedure is valid only when the series in (0.1) and (4.2) have a common domain of convergence in the complex x-plane. Put $p=re^{i\alpha}$, $0<r<1$, $-\pi<\alpha\le\pi$. Then the series in (0.1) converges if and only if $|x|^2<(\operatorname{Log} r)^2+\alpha^2$. The double series in (4.2) are absolutely convergent when $\operatorname{Re} x>\operatorname{Log} r$. Thus, there is a common domain of convergence, and the proof is complete.

Entry 5. *We have $\psi_0(p)=1$, while for $n\ge 1$,*

$$(-1)^{n+1}p(p+1)^{-n}\psi_n(p)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(p+1)^{-k}\psi_k(p). \tag{5.1}$$

Proof. From (0.1),

$$1=\left\{p+\sum_{n=0}^{\infty}\frac{x^n}{n!}\right\}\sum_{n=0}^{\infty}\frac{(-1)^n\psi_n(p)x^n}{n!\,(p+1)^{n+1}}.$$

Equating coefficients on both sides, we find that $\psi_0(p)=1$ and that (5.1) holds for $n\geq 1$.

For $n\geq 1$, the recursion formula (5.1) may be written in the form

$$\psi_n(p)=\sum_{j=0}^{n-1}(-1)^j\binom{n}{j+1}(p+1)^j\psi_{n-j-1}(p), \tag{5.2}$$

where we have set $j=n-k-1$ in (5.1). Note that $\psi_1(p)=1$. By inducting on n, we see that $\psi_n(p)$ is a polynomial in p of degree $n-1$. Thus, after Ramanujan, we write

$$\psi_n(p)=\sum_{k=0}^{n-1}F_{k+1}(n)(-p)^k. \tag{6.1}$$

Entry 6. *Let $1\leq r\leq n$. Then*

(i) $$F_r(n)=F_{n-r+1}(n),$$

(ii) $$\sum_{k=0}^{r-1}\binom{n+k}{k}F_{r-k}(n)=r^n,$$

and

(iii) $$F_r(n)=\sum_{k=0}^{r-1}(-1)^k\binom{n+1}{k}(r-k)^n.$$

Proof of (i). Since $\psi_0(p)=1$, (0.1) yields

$$\frac{e^x-1}{(p+1)(e^x+p)}=\frac{1}{p+1}-\frac{1}{e^x+p}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\psi_n(p)x^n}{n!\,(p+1)^{n+1}}.$$

Replacing x by $-x$ and p by $1/p$, we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\psi_n(p)x^n}{n!\,(p+1)^n}=\frac{e^x-1}{e^x+p}=\sum_{n=1}^{\infty}\frac{p^{n-1}\psi_n(1/p)x^n}{n!\,(p+1)^n}.$$

Equating coefficients of x^n, we find that

$$(-1)^{n-1}\psi_n(p)=p^{n-1}\psi_n\left(\frac{1}{p}\right), \qquad n\geq 1. \tag{6.2}$$

Using (6.1) in (6.2) and equating coefficients of p^{r-1} on both sides, we complete the proof of (i).

Proof of (ii). By (4.1) and (6.1), if $|p|<1$,

$$\sum_{k=1}^{\infty}k^n(-p)^{k-1}=\sum_{j=1}^{n}F_j(n)(-p)^{j-1}\sum_{k=0}^{\infty}\binom{n+k}{k}(-p)^k.$$

Equating the coefficients of p^{r-1} on both sides, we deduce (ii).

Proof of (iii). Again, by (4.1) and (6.1), for $|p| < 1$,

$$\sum_{k=0}^{n-1} F_{k+1}(n)(-p)^k = \sum_{k=0}^{\infty} (k+1)^n(-p)^k \sum_{j=0}^{n+1} \binom{n+1}{j} p^j.$$

Equate coefficients of p^{r-1} on both sides to deduce (iii).

The statement of (ii) in the notebooks, p. 48, is incorrect; replace n by $n+1$ on the left side of (ii). Entry 6(ii) is due to Worpitzky [1] while (iii) is due to Euler [1].

Entry 7. *If* $n \geq 0$ *and* $0 < x < 1 - e^{-2\pi}$, *then*

$$\psi_n(x-1) = \frac{x^{n+1}}{1-x}\left\{\frac{n!}{\left(\operatorname{Log}\dfrac{1}{1-x}\right)^{n+1}} + (-1)^n \sum_{k=n+1}^{\infty} \frac{B_k}{k(k-n-1)!}\left(\operatorname{Log}\frac{1}{1-x}\right)^{k-n-1}\right\}, \tag{7.1}$$

where B_k *denotes the* kth *Bernoulli number.*

This formulation of Entry 7 is not the same as Ramanujan's version. Ramanujan claims that $\psi_n(x-1)$ is the "integral part" of

$$\frac{x^{n+1}}{1-x}\left\{\frac{n!}{\left(\operatorname{Log}\dfrac{1}{1-x}\right)^{n+1}} + (-1)^n \frac{B_{n+1}}{n+1}\right\}. \tag{7.2}$$

Since $\psi_n(x-1)$ is generally not an integer, we are not sure what Ramanujan intends. Perhaps Ramanujan is indicating that the primary contribution to $\psi_n(x-1)$ in (7.1) is (7.2), especially if x is small.

Proof. By (I1), if $0 < x < 2\pi$,

$$\sum_{k=1}^{\infty} e^{-kx} = \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^{k-1}. \tag{7.3}$$

Differentiate both sides of (7.3) n times with respect to x and multiply both sides by $(-1)^n$ to get

$$\sum_{k=1}^{\infty} k^n e^{-kx} = \frac{n!}{x^{n+1}} + (-1)^n \sum_{k=n+1}^{\infty} \frac{B_k}{k(k-n-1)!} x^{k-n-1}. \tag{7.4}$$

Now replace x by $-\operatorname{Log}(1-x)$ in (7.4). We then observe that the left side of (7.4) is $(1-x)x^{-n-1}\psi_n(x-1)$ by Entry 4. This completes the proof of (7.1).

Entry 8. *For* $n \geq 1$, $\psi_n(-1) = n!$ *and* $\psi_n(1) = 2^{n+1}(2^{n+1}-1)B_{n+1}/(n+1)$. *Furthermore,*

$$\begin{aligned}
\psi_0(p) &= \psi_1(p) = 1,\\
\psi_2(p) &= 1 - p,\\
\psi_3(p) &= 1 - 4p + p^2,\\
\psi_4(p) &= 1 - 11p + 11p^2 - p^3,\\
\psi_5(p) &= 1 - 26p + 66p^2 - 26p^3 + p^4,\\
\psi_6(p) &= 1 - 57p + 302p^2 - 302p^3 + 57p^4 - p^5,
\end{aligned}$$

and

$$\psi_7(p) = 1 - 120p + 1191p^2 - 2416p^3 + 1191p^4 - 120p^5 + p^6.$$

Proof. Since $\psi_0(-1) = 1$, the values for $\psi_n(-1)$, $n \geq 1$, follow immediately from (5.2) and induction on n.

Next, by (I1), for $|x| < \pi$,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(1-2^k)B_k x^{k-1}}{k!} &= \frac{1}{e^x - 1} - \frac{2}{e^{2x} - 1}\\
&= \frac{1}{e^x + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n \psi_n(1) x^n}{2^{n+1} n!}.
\end{aligned}$$

The formula for $\psi_n(1)$ now readily follows from comparing coefficients of x^n above.

By Entry 5 and (5.2), we previously had shown that $\psi_0(p) = \psi_1(p) = 1$. To calculate the remaining polynomials, we employ Entry 6(i) and the recursion formula

$$F_k(n) = kF_k(n-1) + (n-k+1)F_{k-1}(n-1), \tag{8.1}$$

where $2 \leq k \leq n$. Ramanujan does not state (8.1), but he indicates that he was in possession of such a formula.

To prove (8.1), we employ Entry 6(iii) to get

$$\begin{aligned}
&kF_k(n-1) + (n-k+1)F_{k-1}(n-1)\\
&= k\sum_{j=0}^{k-1} (-1)^j \binom{n}{j} (k-j)^{n-1} + (n-k+1) \sum_{j=0}^{k-2} (-1)^j \binom{n}{j} (k-j-1)^{n-1}\\
&= k^n + \sum_{j=1}^{k-1} (-1)^j (k-j)^{n-1} \left\{ k\binom{n}{j} - (n-k+1)\binom{n}{j-1} \right\}\\
&= \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j} (k-j)^n = F_k(n).
\end{aligned}$$

Corollary 1. *Let $f(x)$ denote the solution found in Entry 3. Then $f(x)$ is the term independent of n in*

$$\frac{\sum_{k=0}^{\infty} \varphi^{(k)}(x)/n^k}{e^{nh}+p}, \tag{8.2}$$

where it is understood that the series in the numerator above does, indeed, converge.

Proof. Expand $1/(e^{nh}+p)$ by (0.1). Upon the multiplication of the two series in (8.2), the proposed result readily follows.

Corollary 2 is a complete triviality and not worth recording here.

Corollary 3. *If n is even and positive, then $\psi_n(p)$ is divisible by $1-p$.*

Proof. The result follows readily from (6.1) and Entry 6(i).

Corollary 4. *For $|x|<|\text{Log}(-p)|$,*

$$\frac{\cos x+p}{p^2+2p\cos x+1}=\sum_{n=0}^{\infty}\frac{(-1)^n\psi_{2n}(p)x^{2n}}{(2n)!\,(p+1)^{2n+1}}.$$

Corollary 5. *For $|x|<|\text{Log}(-p)|$,*

$$\frac{\sin x}{p^2+2p\cos x+1}=\sum_{n=0}^{\infty}\frac{(-1)^n\psi_{2n+1}(p)x^{2n+1}}{(2n+1)!\,(p+1)^{2n+2}}.$$

Proofs of Corollaries 4 *and* 5. Replacing x by ix in (0.1), we have

$$\frac{1}{e^{ix}+p}=\frac{e^{-ix}+p}{p^2+2p\cos x+1}=\sum_{n=0}^{\infty}\frac{\psi_n(p)(-ix)^n}{n!\,(p+1)^{n+1}}.$$

Equating real and imaginary parts on both sides above, we deduce Corollaries 4 and 5, respectively.

In the hypothesis of Corollary 6, which consists of four parts, Ramanujan attempts to define a sequence of numbers $\{A_n\}$. However, these numbers, as defined by Ramanujan, are not uniquely determined. It is preferable to define A_n by Corollary 6(iv) and then deduce the equality of the hypothesis. Hence, we have taken the liberty of inverting (iv) and the hypothesis below. Thus, put

$$\psi_n(p-1)=\sum_{k=0}^{n-1}A_{n-k}(-p)^k. \tag{8.3}$$

Ramanujan's notation is unfortunate because A_k depends upon n.

For Re $s>1$, the Riemann zeta-function $\zeta(s)$ is defined by $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$; in Ramanujan's notation, $\zeta(k)=S_k$.

Corollary 6. *Let* $1 \le r \le n$. *Then*

(i) $$\sum_{k=1}^{r} \binom{r}{k} A_k = r^n,$$

(ii) $$A_r = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} (r-k)^n,$$

(iii) $A_r/n!$ *is the coefficient of* x^n *in* $(e^x - 1)^r$,

and

(iv) $$\sum_{k=1}^{\infty} (-1)^{k+1} k^n (\zeta(k+1) - 1)$$
$$= (-1)^n + (-1)^n 2^{-n-1} \psi_n(1) + \sum_{k=1}^{n} (-1)^{k+1} A_k \zeta(k+1),$$

where $\psi_n(1)$ *is determined in Entry* 8.

Proof of (i). By (6.1),

$$\psi_n(p-1) = \sum_{k=0}^{n-1} F_{k+1}(n)(1-p)^k. \tag{8.4}$$

Equating coefficients of p^{n-j} in (8.3) and (8.4), we find that, for $1 \le j \le n$,

$$A_j = \sum_{k=n-j}^{n-1} \binom{k}{n-j} F_{k+1}(n).$$

Thus, using Vandermonde's theorem below, we find that

$$\begin{aligned}\sum_{j=1}^{r} \binom{r}{j} A_j &= \sum_{j=1}^{r} \binom{r}{j} \sum_{k=n-j}^{n-1} \binom{k}{n-j} F_{k+1}(n) \\ &= \sum_{k=n-r}^{n-1} F_{k+1}(n) \sum_{j=n-k}^{r} \binom{r}{j} \binom{k}{n-j} \\ &= \sum_{k=n-r}^{n-1} \binom{r+k}{n} F_{k+1}(n) \\ &= \sum_{j=0}^{r-1} \binom{n+j}{n} F_{n+j-r+1}(n) \\ &= \sum_{j=0}^{r-1} \binom{n+j}{j} F_{r-j}(n) = r^n,\end{aligned}$$

where we have employed Entries 6(i) and 6(ii).

Proof of (ii). The proposed formula follows from the inversion of (i). (See Riordan's book [2, pp. 43, 44].)

From (ii) it is seen that $A_r = A_r(n) = r!\, S(n, r)$, where the numbers $S(n, r)$ are Stirling numbers of the second kind. For the definition and basic properties of these numbers, see the work of Abramowitz and Stegun [1, pp. 824, 825] as well as Sections 6–8 in Chapter 3 here. Because (iii) is such a familiar property of Stirling numbers of the second kind and because its proof is similar to that of Example 2, Section 8, of Chapter 3, we omit the proof.

Proof of (iv). Using successively Entry 4, (6.2), and (8.3) below, we find that

$$\sum_{k=1}^{\infty} (-1)^{k+1}k^n(\zeta(k+1)-1) = \sum_{j=2}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^n}{j^{k+1}}$$

$$= \sum_{j=2}^{\infty} \frac{\psi_n(1/j)}{j^2(1+1/j)^{n+1}}$$

$$= (-1)^{n-1}\sum_{j=2}^{\infty} \frac{\psi_n(j)}{(j+1)^{n+1}}$$

$$= (-1)^{n-1}\sum_{j=2}^{\infty} \frac{1}{(j+1)^{n+1}}\sum_{k=0}^{n-1} (-1)^k A_{n-k}(j+1)^k$$

$$= (-1)^{n-1}\sum_{k=0}^{n-1} (-1)^k A_{n-k}\{\zeta(n-k+1)-1-2^{k-n-1}\},$$

from which the desired formula follows with the use of (8.3).

Examples 1 and 2. *We have*

$$\sum_{k=1}^{\infty} \frac{k^5}{2^k} = 1082 \quad \textit{and} \quad \sum_{k=1}^{\infty} \frac{k^5}{3^k} = \frac{273}{4}.$$

Proof. By Entry 4 and (6.2),

$$\sum_{k=1}^{\infty} \frac{k^n}{p^k} = \frac{\psi_n(-1/p)}{p(1-1/p)^{n+1}} = \frac{p\psi_n(-p)}{(p-1)^{n+1}}. \tag{8.5}$$

From Entry 8, $\psi_5(-2) = 541$ and $\psi_5(-3) = 1456$. Putting these values in (8.5), we achieve the two given evaluations.

Entry 9 is simply the definition (I1) of the Bernoulli numbers, and Entry 10 is the expansion given on the right side of (1.3).

Entry 11. *For* $|x| < 2\pi$,

$$\operatorname{Log}\left(\frac{x}{e^x-1}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}B_n x^n}{n\, n!}.$$

Proof. Using (I1), we have, for $|x| < 2\pi$,

$$\operatorname{Log}\left(\frac{x}{e^x - 1}\right) = \int_0^x \left(\frac{1}{t} - \frac{e^t}{e^t - 1}\right) dt = \int_0^x \frac{1}{t}\left(1 - \frac{-t}{e^{-t} - 1}\right) dt$$

$$= \int_0^x \sum_{n=1}^{\infty} \frac{B_n}{n!}(-t)^{n-1}\, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n x^n}{n\, n!}.$$

Entry 12. *For* $|x| < \pi$,

$$\operatorname{Log}\left(\frac{2}{e^x + 1}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2^n - 1) B_n x^n}{n\, n!}.$$

Proof. Observe that

$$\operatorname{Log}\left(\frac{2}{e^x + 1}\right) = \operatorname{Log}\left(\frac{2x}{e^{2x} - 1}\right) - \operatorname{Log}\left(\frac{x}{e^x - 1}\right)$$

and then use Entry 11.

The conclusions of Examples 1–3 below are written as equalities in the notebooks, but Ramanujan clearly realizes that his results are approximations. Put $e^P = 1 + p$, $e^Q = 1 + q$, $e^R = 1 + r$, $e^S = 1 + s$, and $e^T = 1 + t$, where p, q, r, s, and t are to be regarded as small. In the first example below, Ramanujan has incorrectly written $-1/2$ instead of $1/2$ on the right side.

Example 1. *If* $e^P + e^Q + e^R = 2 + e^{P+Q+R}$, *then*

$$\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{12}(P + Q + R) \approx \frac{1}{2}.$$

Proof. In terms of p, q, and r, we are given that $3 + p + q + r = 2 + (1 + p)(1 + q)(1 + r)$, which may be written as

$$-1 = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{e^P - 1} + \frac{1}{e^Q - 1} + \frac{1}{e^R - 1}$$

$$\approx \frac{1}{P} - \frac{1}{2} + \frac{P}{12} + \frac{1}{Q} - \frac{1}{2} + \frac{Q}{12} + \frac{1}{R} - \frac{1}{2} + \frac{R}{12},$$

where we have used (I1) and ignored all terms with powers of P, Q, and R greater than the first. The desired approximation now follows.

Example 2. *If*

$$e^{P+Q+R+S} = \frac{e^P + e^Q + e^R + e^S - 2}{e^{-P} + e^{-Q} + e^{-R} + e^{-S} - 2},$$

then

$$\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+\frac{1}{12}(P+Q+R+S)\approx 0.$$

Proof. The given equality is equivalent to

$$(1+p)(1+q)(1+r)(1+s)=\frac{2+p+q+r+s}{2-p-q-r-s}.$$

Now cross multiply and ignore all products involving p^2, q^2, r^2, and s^2. After a tedious calculation and much simplification, we find that

$$-2\approx\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}.$$

Proceeding as in Example 1, we achieve the sought approximation.

Example 3. *If*

$$2e^{P+Q+R+S+T}$$
$$=\frac{(e^P+e^Q+e^R+e^S+e^T-2)^2-(e^{2P}+e^{2Q}+e^{2R}+e^{2S}+e^{2T}-2)}{e^{-P}+e^{-Q}+e^{-R}+e^{-S}+e^{-T}-2},$$

then

$$\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+\frac{1}{T}+\frac{1}{12}(P+Q+R+S+T)\approx\frac{1}{2}.$$

Proof. The proof is straightforward, very laborious, and along the lines of the proofs of Examples 1 and 2. Rewrite the given equality in terms of p, q, r, s, and t. Cross multiply and ignore all terms involving p^2, q^2, r^2, s^2, and t^2. After a lengthy calculation and considerable cancellation, we arrive at

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}+\frac{1}{t}\approx -2.$$

Now proceed as in Example 1.

Entry 13. *For* $|x|<\pi$,

$$x\cot x=\sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}(2x)^{2n}}{(2n)!}.$$

Entry 14. *For* $|x|<\pi$,

$$x\csc x=\sum_{n=0}^{\infty}\frac{(-1)^n(2-2^{2n})B_{2n}x^{2n}}{(2n)!}.$$

Entry 15. *For* $|x|<\pi/2$,

$$x\tan x=\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}(1-2^{2n})B_{2n}x^{2n}}{(2n)!}.$$

Entries 13–15 are very familiar, and so there is no point in supplying proofs here.

Entry 16. *For* $|x| < \pi$,

$$\operatorname{Log}\left(\frac{x}{\sin x}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{2n}(2x)^{2n}}{(2n)(2n)!}.$$

Proof. For $|x| < \pi$,

$$\operatorname{Log}\left(\frac{x}{\sin x}\right) = \int_0^x \frac{1}{t}(1 - t \cot t)\, dt.$$

Now employ Entry 13.

Entry 17. *For* $|x| < \pi/2$,

$$\operatorname{Log}(\sec x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}(1 - 2^{2n}) B_{2n} x^{2n}}{(2n)(2n)!}.$$

Proof. For $|x| < \pi/2$,

$$\operatorname{Log}(\sec x) = \int_0^x \tan t\, dt.$$

Now employ Entry 15.

Ramanujan now makes three remarks, the first of which is trivial and the second of which is a special case of the third,

$$\frac{B_{2n}}{B_{2n-2h}} \sim \frac{(-1)^h (2n)!}{(2\pi)^{2h}(2n-2h)!}$$

as n tends to ∞, where $0 \le h \le n-1$. This asymptotic formula is a simple consequence of Euler's formula for $\zeta(2n)$; see Entry 25(i).

The following recurrence relation is due to Euler [1].

Entry 18. *Let n be an integer exceeding* 1. *Then*

$$-(2n+1)B_{2n} = \sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k}.$$

Ramanujan's first published paper [4], [15, pp. 1–14] is on Bernoulli numbers. and Section 2 of that paper contains a proof of Entry 18. After Entry 18, Ramanujan lists all of the Bernoulli numbers with index ≤ 38. All of the values are correct and agree with Abramowitz and Stegun's tables [1, p. 810]. The next result is contained in (16) of Ramanujan's paper [4].

Entry 19(i). *For each positive integer n*, $2(2^{2n} - 1)B_{2n}$ *is an integer.*

Proof. We give a somewhat easier proof than that in [4]. By the von Staudt–Clausen theorem, the denominator of B_{2n} is the product of all those primes p such that $(p-1)|2n$. Let p be such an odd prime. Then by Fermat's theorem, $p|(2^{p-1}-1)$. But since $(p-1)|2n$, we have $p|(2^{2n}-1)$, which completes the proof.

Entry 19(ii). *The numerator of B_{2n} is divisible by the largest factor of $2n$ which is relatively prime to the denominator of B_{2n}.*

Entry 19(ii) is contained in (18) of Ramanujan's paper [4] and is originally due to J. C. Adams. (See Uspensky and Heaslet's book [1, p. 261].) In fact, in both Entry 19(ii) and (18) of [4], Ramanujan claims a stronger result, viz., the implied quotient is a prime number. However, this is false; for example, the numerator of B_{22} is $854513 = 11 \cdot 131 \cdot 593$. (See, for example, Wagstaff's table in [1, p. 589].)

Entry 19(iii) is a statement of the von Staudt–Clausen theorem, which we mentioned above and which is (19) of Ramanujan's paper [4].

Entry 20. *For each nonnegative integer n, $(-1)^{n-1}B_{2n}+(-1)^n(1-F_{2n})$ is an integer I_{2n}, where F_{2n} is the sum of the reciprocals of those primes p such that $(p-1)|2n$.*

Of course, Entry 20 is another version of the von Staudt–Clausen theorem. Ramanujan next lists the following values for I_{2n}: $I_{2n}=0$, $0 \le n \le 6$, $I_{14}=1$, $I_{16}=7$, $I_{18}=55$, $I_{20}=529$, $I_{22}=6192$, $I_{24}=86580$, and $I_{26}=1425517$. All of these values are correct.

In Example 1, Ramanujan calculates B_{22}, partly with the aid of the von Staudt–Clausen theorem. However, his reasoning is fallacious because Ramanujan thought that the numerator of B_{22} divided by 11 is prime. We pointed out above that this is false.

Example 2. *The fractional part of B_{200} is 216641/1366530.*

Proof. The given result is a direct consequence of the von Staudt–Clausen theorem.

Several of the results in Ramanujan's first paper [4] are not completely proved or are false. Wagstaff [2] has made a thorough examination of Ramanujan's paper and has given complete proofs of all the correct results in [4].

Entry 21 consists of two tables. The first is a table of primes up to 211. In constructing the table, Ramanujan makes use of the fact that any prime other than 2, 3, and 5 is of the form $p+30n$, where $n \ge 0$ and p is either 7, 11, 13, 17, 19, 23, 29, or 31. The second table lists all primes up to 4969. The numbers at the extreme left of the table give the number of hundreds in the primes immediately following; and the inked vertical strokes divide the hundreds. The table was presumably constructed as follows. First insert the primes (constructed in the first table) up to 211. Then use the fact that any prime other than 2, 3, 5, and 7 is of the form $q + 210n$, where $n \geq 0$ and q is one of the numbers 11, 13, 17, ..., 199, 211. Thus, increase each of these primes already in the table by multiples of 210 until we reach the prime $2\cdot 3\cdot 5\cdot 7\cdot 11 + 1 = 2311$. Then proceed in a similar fashion with the arithmetic progressions $r + 2310n$, where $n \geq 0$ and r is any of the numbers 13, 17, 19, ..., 2309, 2311.

Entry 22. *If n is a natural number, then*

$$2^{2n}(2^{2n} - 1)B_{2n} = 2n \sum_{k=0}^{n-1} \binom{2n-2}{2k} E_{2k}E_{2n-2k-2},$$

where E_j denotes the jth *Euler number.*

Proof. By (I2) and Entry 15,

$$\left\{\sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} x^{2k}}{(2k)!}\right\}^2 = \sec^2 x = \frac{d}{dx}\tan x$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}(1 - 2^{2n})B_{2n} x^{2n-2}}{(2n)(2n-2)!}.$$

Now equate coefficients of x^{2n-2} on both sides to achieve the proposed formula.

Next, Ramanujan records the Euler numbers E_{2n}, $0 \leq n \leq 7$. All values are correct.

Entries 23(i)–(iv) give the well-known partial fraction decompositions of $\cot(\pi x)$, $\tan(\pi x/2)$, $\csc(\pi x)$, and $\sec(\pi x/2)$. Ramanujan has also derived these expansions in Chapter 2, Entry 10, Corollaries 1–3. Entries 24(i)–(iv) offer the familiar partial fraction expansions of $\coth(\pi x)$, $\tanh(\pi x/2)$, $\operatorname{csch}(\pi x)$, and $\operatorname{sech}(\pi x/2)$.

After Entry 24, Ramanujan makes three remarks. In the first he claims that the last digit of E_{4n} is 5 and that $E_{4n-2} + 1$ is divisible by 4, $n \geq 1$. Ramanujan probably discovered these results empirically. The latter is due to Sylvester. Moreover, they are, respectively, special cases of the following congruences

$$E_{4n} \equiv 5 \pmod{60} \quad \text{and} \quad E_{4n-2} \equiv -1 \pmod{60},$$

normally attributed to Stern. (See Nielsen's text [5, p. 261].)

The second remark is a special case of the third, namely,

$$\frac{E_{2n+2h}}{E_{2n}} \sim \frac{(-1)^h 2^{2h}(2n+2h)!}{\pi^{2h}(2n)!}$$

as n tends to ∞, where h is a nonnegative integer. This asymptotic formula is an easy consequence of Entry 25(iv) below.

For Re $s > 1$, let $\lambda(s) = \sum_{k=0}^{\infty} (2k+1)^{-s}$, and for Re $s > 0$, define $\eta(s) = \sum_{k=1}^{\infty} (-1)^{k+1} k^{-s}$ and $L(s) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-s}$.

Entry 25. *If n is a positive integer, then*

(i) $$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n},$$

(ii) $$\lambda(2n) = \frac{(-1)^{n-1}(1-2^{-2n})(2\pi)^{2n}}{2(2n)!} B_{2n},$$

(iii) $$\eta(2n) = \frac{(-1)^{n-1}(1-2^{1-2n})(2\pi)^{2n}}{2(2n)!} B_{2n},$$

and

(iv) $$L(2n-1) = \frac{(-1)^{n-1}(\pi/2)^{2n-1}}{2(2n-2)!} E_{2n-2}.$$

The first equality is Euler's very famous formula for $\zeta(2n)$. For an interesting account of Euler's discovery of this formula, see Ayoub's paper [2]. A proof of (i) that uses only elementary calculus is found in Berndt's article [2], which also contains references to several other proofs. Observe that $\lambda(s) = (1 - 2^{-s})\zeta(s)$ and that $\eta(s) = (1 - 2^{1-s})\zeta(s)$, and so (ii) and (iii) both follow from (i). Formula (iv) is also well known and is an illustration of the following fact. If χ is an odd character, then the associated Dirichlet L-function can be explicitly evaluated at odd, positive integral values; if χ is even, then the associated L-function can be determined at even, positive integral values. See Berndt and Schoenfeld's work [1] for a verification of this remark.

Ramanujan next uses (i) and (iii) to define Bernoulli numbers for any index. Thus, for any real number s, he defines

$$B_s^* = \frac{2\Gamma(s+1)}{(2\pi)^s} \zeta(s). \tag{25.1}$$

(We now employ Ramanujan's convention for the ordinary Bernoulli numbers.) Note that $B_{2n+1}^* \neq 0$, $n \geq 1$, in Ramanujan's definition, which conflicts with (I1). It is curious that Euler [1], [7, p. 350] also defined Bernoulli numbers of arbitrary index by (25.1) but apparently made no significant use of this idea.

Similarly, Euler numbers of arbitrary real index s may be defined by interpolating (iv) above. Thus, Ramanujan defines E_s^* by

$$E_s^* = \frac{2\Gamma(s)}{(\pi/2)^s} L(s),$$

for any real number s. Observe that $E_{2n+1}^* = (-1)^n E_{2n}$, where n is a nonnegative integer and E_{2n} denotes the $2n$th Euler number. (Ramanujan did not adjoin a suffix * to B_s or E_s; we have done so to distinguish the ordinary Bernoulli and Euler numbers from Ramanujan's extensions.) The next two corollaries give numerical examples.

Corollary 1. $B_1^* = \infty$, $B_{3/2}^* = \dfrac{3\zeta(3/2)}{4\pi\sqrt{2}}$, *and* $B_3^* = \dfrac{3\zeta(3)}{2\pi^3}$.

Corollary 2. $B_0^* = -1$, $B_{1/2}^* = -(1+1/\sqrt{2})\eta(1/2)$, $E_0^* = \infty$, $E_{1/2}^* = 2\sqrt{2}L(1/2)$, *and* $E_2^* = 8L(2)/\pi^2$.

Proof. The value for B_0^* arises from the extension (25.1) and the fact that $\zeta(0) = -1/2$. The value for E_0^* arises from the fact that $\Gamma(s)$ has a simple pole at $s = 0$. All other tabulated values are easily verified.

Let

$$f(a) = \sum_{k=1}^{\infty} (a + bk)^{-n},$$

where a, $b > 0$ and $n > 1$. Note that $f(a) = b^{-n}\zeta(n, a/b) - a^{-n}$, where $\zeta(s, \alpha)$ denotes Hurwitz's zeta-function.

Entry 26. *As b/a tends to 0,*

$$a^n f(a) \sim \frac{a}{b(n-1)} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{n-1} \binom{n+2k-2}{2k} \left(\frac{b}{a}\right)^{2k-1}. \tag{26.1}$$

Proof. A simple calculation shows that $f(a-b) - f(a) = a^{-n}$. In the notation of Entry 1(i), $h = -b$ and $\varphi(a) = 1/\{a^{n-1}b(n-1)\}$. If Entry 1(i) were applicable, we could readily deduce (26.1) with the asymptotic sign $\sim$ replaced by an equality sign. But, the series on the right side of (26.1) does not converge. However, by appealing to the theorem in Nörlund's text [2, pp. 58–60] that we mentioned after the proof of Entry 1(ii), we can conclude that the right side of (26.1) represents the function $a^n f(a)$ asymptotically as b/a tends to 0.

Example. *For $n > 1$, as the positive integer r tends to ∞,*

$$\zeta(n) \sim \sum_{k=1}^{r-1} k^{-n} + \frac{1}{(n-1)r^{n-1}} + \frac{1}{2r^n} + \sum_{k=1}^{\infty} \frac{B_{2k}}{n-1} \binom{n+2k-2}{2k} r^{-n-2k+1}. \tag{26.2}$$

Proof. Apply Entry 26 with $a = r$ and $b = 1$. Thus, as r tends to ∞,

$$\sum_{k=1}^{\infty} (r+k)^{-n} \sim \frac{1}{(n-1)r^{n-1}} - \frac{1}{2r^n} + \sum_{k=1}^{\infty} \frac{B_{2k}}{n-1} \binom{n+2k-2}{2k} r^{-n-2k+1}.$$

Adding $\sum_{k=1}^{r} k^{-n}$ to both sides above, we obtain (26.2).

Asymptotic series like those in Entry 26 and the Example above were initially found by Euler. See Bromwich's book [1, pp. 324–329] for a very complete discussion of such asymptotic series and their applications to numerical calculation. Indeed, like Euler, Ramanujan employs (26.2) to calculate $\zeta(n)$, where n is a positive integer with $2 \leq n \leq 10$, to the tenth decimal place and provides the following table.

n	$\zeta(n)$	$1/B_n^*$
2	1.6449340668	6
3	1.2020569031	17.19624
4	1.0823232337	30
5	1.0369277551	39.34953
6	1.0173430620	42
7	1.0083492774	38.03538
8	1.0040773562	30
9	1.0020083928	20.98719
10	1.0009945781	13.2

For $\zeta(3)$, the tenth decimal place should be 2; for $\zeta(10)$, the ninth decimal place should be 5. Euler used this same method to calculate $\zeta(n)$, $2 \leq n \leq 16$, to 18 decimal places. (See Bromwich's book [1, p. 326].) A different method was used by Legendre [1, p. 432] to calculate $\zeta(n)$, $2 \leq n \leq 35$, to 16 decimal places. Stieltjes [1], using Legendre's method, calculated $\zeta(n)$, $2 \leq n \leq 70$, to 32 decimal places. For even n, the values of $1/B_n^*$ above are determined from Ramanujan's table of Bernoulli numbers. For odd n, Ramanujan employs (25.1) and his previously determined values of $\zeta(n)$. In the far right column above, the last recorded digit for $1/B_3^*$ should be 3; the last digit for $1/B_7^*$ should be 6; and the last two digits for $1/B_9^*$ should be 20.

Corollary 1. *The Riemann zeta-function has a simple pole at $s = 1$ with residue 1, and the constant term in the Laurent expansion of $\zeta(s)$ about $s = 1$ is Euler's constant γ.*

Of course, Corollary 1 is very well known. (For example, see the books of Landau [1, p. 164] or Titchmarsh [3, p. 16].) Ramanujan's wording for Corollary 1 is characteristically distinct: "$n\ S_{n+1} = 1$ if $n = 0$ and $S_{n+1} - 1/n = 0.577$ nearly." In the sketch following Corollary 1, Ramanujan gives the first three terms of the asymptotic series

$$\gamma \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n}. \tag{26.3}$$

The symbol $\sim$ here means that the first n terms from the series on the right side approximate the left side with an error less than the magnitude of the $(n+1)$th term. In fact, this asymptotic series for γ was first discovered by

Euler and used by him to calculate γ. (See Bromwich's book [1, pp. 324–325].) It is curious that Ramanujan's approximation 0.577 to γ is better than any that can be gotten from Euler's series. Bromwich [1, p. 325] points out that the best approximation 0.5790 is achieved by taking four terms on the right side of (26.3). If we take just three terms and average the two approximations, we get the mean approximation 0.5770. Perhaps this is how Ramanujan reasoned, or possibly he calculated γ by using the Euler–Maclaurin summation formula to approximate a partial sum of the harmonic series.

Corollary 2 is merely a reformulation of the latter part of Corollary 1 in terms of Ramanujan's generalization (25.1) of Bernoulli numbers.

Entry 27. *Suppose that* $|a_p| \leq p^{-c}$ *for each prime* p *and for some constant* $c > 1$. *Then*

$$\prod_p (1 - a_p)^{-1} = 1 + \sum_{\substack{n=2 \\ n = p_1 \ldots p_k}}^{\infty} a_{p_1} \ldots a_{p_k},$$

where the product on the left is over all primes p, *and where the suffixes on the right side are the (not necessarily distinct) primes in the canonical factorization of* n.

Proof. Expand the product on the left side and use the unique factorization theorem.

Entry 28. *For* Re $s > 1$,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

Entry 28 is the familiar Euler product for $\zeta(s)$ and, in fact, is a special instance of Entry 27. The next two corollaries are simple consequences of Entry 28.

Corollary 1. *For* Re $s > 1$,

$$\prod_p (1 + p^{-s}) = \frac{\zeta(s)}{\zeta(2s)}.$$

Corollary 2. *For* Re $s > 1$,

$$\prod_p \left(\frac{1 + p^{-s}}{1 - p^{-s}}\right) = \frac{\zeta^2(s)}{\zeta(2s)}.$$

Corollary 3. *For* Re $s > 1$,

$$T(s) \equiv \sum_n n^{-s} = \frac{\zeta^2(s) - \zeta(2s)}{2\zeta(s)},$$

where the sum on the left is over all positive integers which have an odd number of prime factors in their canonical factorizations.

Proof. By Entry 28 and Corollary 1,

$$\zeta(s)-\frac{\zeta(2s)}{\zeta(s)}=\prod_p(1-p^{-s})^{-1}-\prod_p(1+p^{-s})^{-1},$$

from which the desired result follows.

Examples 1(i), (ii), and (iii) record the familiar results $\zeta(2)=\pi^2/6$, $L(3)=\pi^3/32$, and $\lambda(4)=\pi^4/96$, respectively, deducible from Entry 25.

Example 2. *We have*

(i) $$\prod_p\left(\frac{1+p^{-2}}{1-p^{-2}}\right)=\frac{5}{2}$$

and

(ii) $$\prod_p(1+p^{-4})=\frac{105}{\pi^4}.$$

Proof. Equality (i) follows from Corollary 2, and (ii) follows from Corollary 1.

Example 3. *We have* $T(2)=\pi^2/20$ *and* $T(4)=\pi^4/1260$, *where* $T(s)$ *is given in Corollary* 3.

Proof. The proposed values follow from Corollary 3 and Entry 25(i).

Corollary 4. *For* Re $s>1$,

$$L(s)=\prod_p\left(1-\sin\left(\frac{\pi p}{2}\right)p^{-s}\right)^{-1}$$

Corollary 4 is simply the Euler product for $L(s)$, and is an instance of Entry 27. Corollary 5 below is a well-known result arising from the logarithmic derivative of the Euler product in Entry 28.

Corollary 5. *For* Re $s>1$,

$$\sum_{k=1}^{\infty}k^{-s}\operatorname{Log}k=\zeta(s)\sum_p\frac{\operatorname{Log}p}{p^s-1}.$$

Example. *The series* $\sum_p \sin(\pi p/2)/p$ *converges.*

Although the result above is a special case of a well-known theorem in the theory of L-functions (see Landau's text [1, pp. 446–449]), its proof is considerably deeper than the other results in Section 28. Ramanujan supplies no hint of how he deduced this result.

Entry 29. *For* $|x| < 1$,

$$\prod_{k=1}^{\infty} (1 - x^{p_k})^{-1} = 1 + \sum_{k=1}^{\infty} \frac{x^{p_1 + p_2 + \cdots + p_k}}{(1-x)(1-x^2)\cdots(1-x^k)}, \tag{29.1}$$

where $p_1, p_2, \ldots$ *denote the primes in ascending order.*

Entry 29, in fact, is canceled by Ramanujan. Let c_n and d_n, $2 \le n < \infty$, denote the coefficients of x^n on the left and right sides, respectively, of (29.1). Then, quite amazingly, $c_n = d_n$ for $2 \le n \le 20$. But $c_{21} = 30$ and $d_{21} = 31$. Thus, as indicated by Ramanujan, (29.1) is false.

G. E. Andrews [3] has discussed Entry 29 and has posed the following problem. Define a "Ramanujan pair" to be a pair of infinite, increasing sequences $\{a_k\}$ and $\{b_k\}$ such that

$$\prod_{k=1}^{\infty} (1 - x^{a_k})^{-1} = 1 + \sum_{k=1}^{\infty} \frac{x^{b_1 + b_2 + \cdots + b_k}}{(1-x)(1-x^2)\cdots(1-x^k)}. \tag{29.2}$$

Had Ramanujan been correct, $\{a_k\} = \{b_k\} = \{p_k\}$ would have been a Ramanujan pair. If $a_k = m + k - 1$ and $b_k = m$, $1 \le k < \infty$, where m is a positive integer, then (29.2) is valid by a theorem of Euler. Also if

$$a_k = \begin{cases} m + k - 1, & 1 \le k \le m, \\ 2k - 1, & k > m, \end{cases}$$

and $b_k = m + k - 1$, $1 \le k < \infty$, (29.2) is satisfied by another identity due to Euler. If

$$\{a_k\} = \{m > 0\colon m \equiv \pm 1 \pmod 5\}$$

and $b_k = 2k - 1$, $1 \le k < \infty$, or if

$$\{a_k\} = \{m > 0\colon m \equiv \pm 2 \pmod 5\}$$

and $b_k = 2k$, $1 \le k < \infty$, (29.2) is again valid. The equalities in these two cases are the Rogers–Ramanujan identities, which are found in Chapter 16. Andrews conjectured that these four pairs exhaust all possibilities. However, Hirschhorn [1] found two additional Ramanujan pairs, and Blecksmith, Brillhart, and Gerst [1] discovered four more Ramanujan pairs. Blecksmith [1] has thoroughly examined this problem, and his computer assisted results strongly suggest that no further Ramanujan pairs exist.

Acreman and Loxton [1] have examined Andrews' problem from another point of view. For what they call regularly varying sequences $\{a_k\}$ and $\{b_k\}$, Acreman and Loxton derive asymptotic formulas for the coefficients of the power series on both sides of (29.2). A comparison of the two asymptotic formulas shows that severe restrictions are placed on the sequences $\{a_k\}$ and $\{b_k\}$ in order for them to be a Ramanujan pair. Furthermore, it is shown that the known Ramanujan pairs correspond to the known values of the dilogarithm, which is defined by (6.1) in Chapter 9.

Entry 30. *Suppose that* $|a_p| \leq p^{-c}$ *for each prime p and for some constant* $c > 1$. *Then*

$$\prod_p (1 + a_p) = 1 + \sum_{\substack{n \\ n = p_1 \dots p_k}} a_{p_1} \dots a_{p_k}.$$

The sum on the right side is over all squarefree integers $n = p_1 \dots p_k$, *where* $p_1, \dots, p_k$ *are distinct primes.*

Proof. Expand the product on the left side above.

Corollary 1. *For* Re $s > 1$,

$$\sum_{\substack{n=1 \\ n \, squarefree}}^{\infty} n^{-s} = \frac{\zeta(s)}{\zeta(2s)}.$$

Proof. Let $a_p = p^{-s}$ in Entry 30 and apply Corollary 1 of Entry 28.

Corollary 2. *For* Re $s > 1$,

$$\sum_n n^{-s} = \frac{\zeta^2(s) - \zeta(2s)}{2\zeta(s)\zeta(2s)},$$

where the sum on the left is over all squarefree integers n which contain an odd number of prime factors.

Proof. By Entry 28 and Corollary 1 of Entry 28,

$$\frac{\zeta(s)}{\zeta(2s)} - \frac{1}{\zeta(s)} = \prod_p (1 + p^{-s}) - \prod_p (1 - p^{-s}),$$

from which the desired equality follows.

Corollary 3. *For* Re $s > 1$,

$$\sum_{\substack{n=1 \\ n \, not \, squarefree}}^{\infty} n^{-s} = \frac{\zeta(s)(\zeta(2s) - 1)}{\zeta(2s)}.$$

Proof. By Corollary 1, the series on the left side is $\zeta(s) - \zeta(s)/\zeta(2s)$.

Entry 27, Entry 28, Corollaries 1 and 3 and Examples 2(ii) and 3 in Section 28, and Entry 30 and its first three corollaries are found in Ramanujan's fourth published paper [7], [15, pp. 20–21].

Corollary 1. *The sum of the reciprocals of the primes diverges.*

As is well known, Corollary 1 is due to Euler, and Ramanujan's proof is similar to Euler's proof, which can be found in Ayoub's text [1, p. 6].

Corollary 2. $\lim_{s\to 1+}\{\text{Log}(s-1)+\sum_p p^{-s}\}$ *exists.*

Proof. By Entry 28, for $s>1$,

$$\text{Log}\,\zeta(s)=\sum_p \frac{1}{p^s}+\sum_p \sum_{k=2}^{\infty} \frac{1}{kp^{ks}}.$$

But by Corollary 1 of Entry 26, $\text{Log}\,\zeta(s) \sim -\text{Log}(s-1)$ as s tends to 1 from the right. The sought result now follows.

Corollary 3. *If* p_n *denotes the* nth *prime, then* $p_n/n - \text{Log}\, n$ *tends to a limit as* n *tends to* ∞.

Ramanujan has rightly struck Corollary 3 out, for (see Landau's book [1, p. 215])

$$\frac{p_n}{n}=\text{Log}\, n+\text{Log Log}\, n+O(1)$$

as n tends to ∞.

CHAPTER 6

Ramanujan's Theory of Divergent Series

In a letter written to A. Holmboe on January 16, 1826, Abel [3] declared that "Divergent series are in general deadly, and it is shameful that anyone dare to base any proof on them." This admonition would have been vehemently debated by Ramanujan. Much like Euler, Ramanujan employed divergent series in a variety of ways to establish a diversity of results, most of them valid but a few not so. Divergent series are copious throughout Ramanujan's notebooks, but especially in Chapter 6 of the second notebook, or in Chapter 8 of the first notebook. Since Ramanujan always uses equality signs in stating identities that involve one or more divergent series, one might be led to believe that Ramanujan probably made no distinction between convergent and divergent series. However, the occasional discourse in Chapter 6 is firm evidence that Ramanujan made such a distinction.

The life of Ramanujan and his mathematics have frequently been enshrouded in an aura of mystery. In particular, Ramanujan's theory of divergent series in Chapter 6 has normally been ensconced in mysticism. However, as we shall see in the sequel, Ramanujan's ideas on this subject are not as strange or as deep as we may have been led to believe. Ramanujan's theory focuses upon the "constant" of a series. This constant is rather imprecisely defined by Ramanujan, but Hardy [15, Chapter 12] has removed its fuzziness and ambiguities. Ramanujan constructs a very tenuous theory based upon his concept of a constant. We shall describe Ramanujan's theory, but readers should keep in mind that his findings frequently lead to incorrect results and cannot be properly described as theorems. Nonetheless, we think it is important to realize how Ramanujan arrived at his results, and so we frequently shall also relate Ramanujan's arguments, even though they are not mathematically rigorous. In fact, some of Ramanujan's "proofs" are sketched in Chapter 6.

As usual, $\zeta(s)=\sum_{k=1}^{\infty} k^{-s}$, Re $s>1$, denotes the Riemann zeta-function. When we write $\sum f(k)$, the limits will always be assumed to be 1 and ∞.

Ramanujan's theory of divergent series emanates from the Euler–Maclaurin summation formula (I3). In Entry 1, Ramanujan states the following special case of (I3):

$$\sum_{k=1}^{x} f(k) = c + \int_0^x f(t)\,dt + \tfrac{1}{2}f(x) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x). \tag{1.1}$$

The limits on the integral are not specified by Ramanujan, but presumably, from his exposition following Entry 1, they are as given above. It is tacitly assumed by Ramanujan that f is such that R_n, defined by (I4), tends to 0 as n tends to ∞. Comparing (I3) and (1.1), we further see that

$$c = -\tfrac{1}{2}f(0) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0). \tag{1.2}$$

Ramanujan calls c the "constant" of the series $\sum f(k)$, which may converge or diverge. He claims that the constant of a series "is like the centre of gravity of a body" (p. 63).

Before proceeding further, we shall make several comments about (1.1) and (1.2). Ramanujan's definition of c is normalized in the sense that, in (I3), α is always taken to be 0. Thus, for example, when examining partial sums of the harmonic series, we should let $f(t)=1/(t+1)$ rather than $f(t)=1/t$.

Secondly, it is a trivial consequence of Euler's formula for $\zeta(2n)$, which is Entry 25(i) in Chapter 5, that

$$|B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}}$$

as n tends to ∞. Thus, in general, the infinite series on the right sides of (1.1) and (1.2) do not converge. However, in many instances the infinite series on the right side of (1.1) is an asymptotic series as x tends to ∞. The infinite series in (1.2) is frequently semiconvergent in the sense that the error (in absolute value) made in terminating the series with the n^{th} term is numerically less (in absolute value) than the $(n+1)^{\text{th}}$ term. See, for example, Bromwich's book [1, articles 106, 107] and Hardy's treatise [5, p. 328] for conditions under which this is the case. When an asymptotic series arises in (1.1) or a semiconvergent series arises in (1.2), we shall use the symbol $\sim$ instead of an equality sign.

When $\sum f(k)$ diverges, the constant of this series is normally the constant in the asymptotic expansion of $\sum_{k=1}^{x} f(k)$ as x tends to ∞. For example, the constant of $\sum 1/k$ is γ. Barnes [1], [2], [3] and Hardy [5], [18, pp. 393–427] have used the constants of asymptotic series for certain sums to define certain zeta-functions and gamma functions.

If $\sum f(k)$ converges, we would like the constant to be the value of the sum of the series. However, in Ramanujan's definition of c, this is hardly ever the case.

The difficulties in Ramanujan's definition of a constant for a series have been overcome by Hardy [15, Chapter 13]. Write (I3), with $\alpha = 0$ and $\beta = x$, in the form

$$\sum_{k=1}^{x} f(k) = \int_a^x f(t)\,dt + \tfrac{1}{2}f(x) + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x) - \int_x^\infty P_{2n+1}(t) f^{(2n+1)}(t)\,dt + C_n,$$

where, for $n \geq 0$,

$$C_n = \int_0^a f(t)\,dt - \tfrac{1}{2}f(0) - \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0) + \int_0^\infty P_{2n+1}(t) f^{(2n+1)}(t)\,dt. \tag{1.3}$$

It is assumed that all indicated integrals exist. The constant C_n does not depend upon n, and so we may write $C_n = C$. Hardy [15, p. 327] calls C the Euler–Maclaurin constant of f. Of course, C depends upon the parameter a. The introduction of the parameter a allows more flexibility and enables one to always obtain the "correct" constant; usually, there is a certain value of a which is more natural than other values. If $\sum f(k)$ converges, then normally we would take $a = \infty$. Although the concept of the constant of a series has been made precise, Ramanujan's concomitant theory cannot always be made rigorous.

Example 1. *The constant of* $\sum 1$ *is* $-\frac{1}{2}$.

Proof. Let $f(t) \equiv 1$ in (1.2).

We remark that the Abel and Cesaro sums of $\sum 1$ are both ∞. (See Hardy's book [15, p. 9].)

Example 2. *The constant of* $\sum k$ *is* $-\frac{1}{12}$.

Proof. Set $f(t) = t$ in (1.2) and use the value $B_2 = \frac{1}{6}$.

It is curious that Ramanujan reasons quite differently in Example 2. He writes

$$\begin{aligned} c &= 1 + 2 + 3 + 4 + \dots, \\ 4c &= \quad\; 4 \quad\; + 8 + \dots, \end{aligned}$$

and so

$$-3c = 1 - 2 + 3 - 4 + \cdots = \frac{1}{(1+1)^2} = \frac{1}{4}. \tag{1.4}$$

Hence, $c = -\frac{1}{12}$. Note that in (1.4) Ramanujan is finding the Abel sum of $\sum(-1)^{k+1}k$. Both Example 2 and (1.4) were communicated by Ramanujan [15, p. 351] in his first letter to Hardy and were discussed by Watson [1].

The constants in Examples 1 and 2 can also be determined from (1.3).

Entry 2 is simply another form of the Euler–Maclaurin summation formula.

Entry 3 is used only to make a definition. Putting $\varphi(x) = \sum_{k=1}^{x} f(k)$, $x \geq 1$, Ramanujan defines $\varphi(-x)$ by $\varphi(-x) = -\sum_{k=1}^{x} f(-k+1)$. For example, if $f(t) = t$, then $\varphi(5) = 15$ and $\varphi(-5) = 10$.

For some of Ramanujan's results in Chapter 6 it is more natural to employ a somewhat different definition of φ. Define $\varphi(x)$ to be a solution of the difference equation

$$\varphi(x) - \varphi(x-1) = f(x). \tag{3.1}$$

Of course, φ is not uniquely defined by (3.1). For example, if φ is any solution of (3.1), the sum of φ and a function with period 1 is also a solution. However, there are two important instances when additional assumptions yield unique solutions. Suppose first that f is a polynomial and we desire φ also to be a polynomial. Then φ is determined up to an additive constant which can be uniquely determined by a boundary condition. Secondly, suppose that $f(x)$ tends to 0 as x tends to ∞, and we wish $\varphi(x)$ to tend to c as x tends to ∞. Then

$$\varphi(x) = c - \sum_{k=1}^{\infty} f(x+k), \tag{3.2}$$

provided that this series converges.

In Entry 4(i), Ramanujan discusses sums with a fractional number of terms, which, if we interpret him properly, we would call partial sums. What Ramanujan seems to be saying is as follows. Suppose that we want to sum $\sum_{k=1}^{x} f(k)$, a sum with a fractional number of terms. Write

$$\sum_{k=1}^{x} f(k) = \sum_{k=1}^{n} f(k) - \sum_{k=x+1}^{n} f(k),$$

where n is "large." Then, in Ramanujan's language, find the constant of $\sum_{k=1}^{n} f(k)$ as n tends to ∞.

Entry 4(ii). *If h and n are positive integers with $h > n$ and if f is analytic for all real numbers, then*

$$\varphi(h) = \varphi(n) - \sum_{k=1}^{n} f(k+h) + \sum_{j=0}^{\infty} \sum_{k=1}^{h} \frac{k^j f^{(j)}(n)}{j!}. \tag{4.1}$$

Proof. The right side of (4.1) is

$$\sum_{k=1}^{n} f(k) - \sum_{k=h+1}^{n+h} f(k) + \sum_{k=1}^{h} \sum_{j=0}^{\infty} \frac{f^{(j)}(n)}{j!} k^j$$
$$= \sum_{k=1}^{n} f(k) - \sum_{k=h+1}^{n+h} f(k) + \sum_{k=1}^{h} f(n+k) = \sum_{k=1}^{h} f(k).$$

Example 1. *Let h and n denote positive integers with h fixed. Then as n tends to* ∞,

$$\sum_{k=1}^{h} \frac{1}{k} \sim \gamma + \operatorname{Log} n - \sum_{k=1}^{n-h} \frac{1}{k+h},$$

where γ *denotes Euler's constant.*

Example 1 is an illustration of Entry 4(i) and is simply a consequence of (1.4) in Chapter 2.

Example 2. *If h is not a negative integer, then*

$$\Gamma(h+1) = \lim_{n\to\infty} \frac{n^h n!}{(h+1)(h+2)\cdots(h+n)}.$$

This is, of course, a well-known product formula for the Γ-function. Ramanujan actually assumes that h is a nonnegative integer, and his proof is straightforward.

Perhaps Ramanujan has something more profound in mind, but Entry 4(iii) offers the trivial identity

$$\varphi(h) = \sum_{k=1}^{\infty} x^k f(k) - \sum_{k=1}^{\infty} x^{k+h} f(k+h),$$

where the notation of Entry 3 is employed and $f(t)$ has been replaced by $x^t f(t)$.

Ramanujan commences Section 5 by defining a series to be *corrected* if its constant is subtracted from it. It is difficult to describe the remainder of Section 5 in mathematically precise language. Ramanujan now considers $\varphi'(x)$. If we strictly adhere to the definition $\varphi(x) = \sum_{k \le x} f(k)$, the proposed study is not very interesting. A different interpretation must be given. In many instances, Ramanujan evidently intends $\varphi'(x)$ to mean the derivative of an asymptotic equivalent to $\varphi(x)$ as x tends to ∞. Or, possibly, we must require $\varphi(x)$ to be an appropriate solution of (3.1). In the formal arguments below we use equality signs; in some particular instances, asymptotic notation must be employed instead.

First, after (3.2), Ramanujan sets

$$\varphi(x) = c - \sum_{k=1}^{\infty} f(x+k),$$

and so

$$\varphi'(x) = -\sum_{k=1}^{\infty} f'(x+k). \tag{5.1}$$

If c' is the constant of the derived series $\sum f'(k)$, then from (5.1) we can also conclude that

$$\varphi'(x) = \sum_{k=1}^{x} f'(k) - c'. \tag{5.2}$$

We emphasize that these arguments are not rigorous.

Example 1. *Let* $\varphi(x)=\sum_{n\leq x} 1/n$. *Then*

$$\varphi'(x) \sim \sum_{k=1}^{\infty} (x+k)^{-2} \tag{5.3}$$

as x *tends to* ∞.

Since, by (1.4) in Chapter 2, $\varphi(x) \sim \operatorname{Log} x$ as x tends to ∞, the function $\varphi'(x)$ on the left side of (5.3) is more appropriately replaced by $1/x$. By Ramanujan's reasoning, (5.3) is an immediate consequence of (5.1). However, (5.3) can be established rigorously. In fact, it is a direct consequence of Entry 26 of Chapter 5. If we define $\varphi(x)$ *for all real* x by the left side of (5.4) below, then, by (5.5), (5.3) is valid with an equality sign.

Example 2. *If* x *is a nonnegative integer, then*

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = -\gamma + \sum_{n\leq x} \frac{1}{n}, \tag{5.4}$$

where γ *denotes Euler's constant.*

Since $\operatorname{Log} \Gamma(x+1) = \sum_{k=1}^{x} \operatorname{Log} k$, Ramanujan obviously established (5.4) by appealing to (5.2), since γ is the constant for the derived series $\sum 1/k$. However, the equality (5.4) can be rigorously and immediately deduced from the well-known relation (see Whittaker and Watson's treatise [1, p. 247])

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z+k}\right), \tag{5.5}$$

which is valid for those complex numbers z that are not negative integers.

Examples 3–6 involve special instances of the sum

$$S_n(x) = \sum_{k\leq x} k^n. \tag{5.6}$$

In particular, if n and x are positive integers and $B_k(x)$ denotes the kth Bernoulli polynomial, then, from Knopp's text [1, p. 526],

$$S_n(x) = \frac{B_{n+1}(x+1) - B_{n+1}}{n+1}. \tag{5.7}$$

Suppose, however, that we *define* $S_n(x)$ by (5.7). Since (see Abramowitz and Stegun's compendium [1, p. 804])

$$B_k(x+1) - B_k(x) = kx^{k-1}, \qquad k \geq 0,$$

$\varphi(x) = S_n(x)$ satisfies (3.1) with $\varphi(0) = 0$ and $f(x) = x^n$ for every real number x. Thus, in Examples 4 and 5 which follow, we shall define $S_n(x)$ by the right side of (5.7) for all x.

Example 3. *For* $x \geq 0$,

$$\int_0^x S_{-1}(t)\,dt = \operatorname{Log}\Gamma(x+1) + \gamma x. \tag{5.8}$$

Proof. If we use (5.6) to define $S_{-1}(x)$ and integrate by parts, we find that

$$\begin{aligned}\int_0^x S_{-1}(t)\,dt &= S_{-1}(t)t\Big|_0^x - \int_0^x t\,dS_{-1}(t)\\ &= xS_{-1}(x) - \sum_{k\leq x} k\cdot\frac{1}{k}\\ &= xS_{-1}(x) - [x]\\ &= x\operatorname{Log}x + (\gamma-1)x + O(1),\end{aligned} \tag{5.9}$$

as x tends to ∞, where we have used (1.4) in Chapter 2.

By Stirling's formula for $\operatorname{Log}\Gamma(x+1)$, which is found in Chapter 7, Entry 23,

$$\begin{aligned}\operatorname{Log}\Gamma(x+1) + \gamma x &\sim (x+\tfrac{1}{2})\operatorname{Log}(x+1) - x + \gamma x + O(1)\\ &= (x+\tfrac{1}{2})\operatorname{Log}x + (\gamma-1)x + O(1),\end{aligned} \tag{5.10}$$

as x tends to ∞. Clearly, (5.9) and (5.10) are incompatible. Thus, with this interpretation of $S_{-1}(x)$, (5.8) is not even valid asymptotically.

Possibly, Ramanujan achieved his evaluation by integrating the equality of (5.4). The fault with this approach stems from the fact that (5.4) is valid only for positive integral x, as can be seen from (5.5).

However, motivated by (5.4), suppose that we define $S_{-1}(x)$ by

$$S_{-1}(x) = \gamma + \frac{\Gamma'(x+1)}{\Gamma(x+1)}. \tag{5.11}$$

Then a simple calculation shows that $\varphi(x) = S_{-1}(x)$ is a solution of (3.1) with $f(x) = 1/x$. With this definition of $S_{-1}(x)$, Ramanujan's result is readily verified.

Example 4. *If* $S_{13}(x)$ *is defined by* (5.7) *with* $n = 13$, *then*

$$\int_0^x S_{13}(t)\,dt = \frac{S_{14}(x)}{14} - \frac{x}{12}. \tag{5.12}$$

Proof. By (5.7),

$$\begin{aligned}\int_0^x S_{13}(t)\,dt &= \int_0^x \frac{B_{14}(t+1) - B_{14}}{14}\,dt = \int_0^x \frac{B'_{15}(t+1)}{15\cdot 14}\,dt - \frac{x}{12}\\ &= \frac{B_{15}(x+1) - B_{15}(1)}{15\cdot 14} - \frac{x}{12} = \frac{1}{14}S_{14}(x) - \frac{x}{12}.\end{aligned}$$

In the calculation above we used the facts (see Abramowitz and Stegun's tables [1, pp. 804, 810]) $B_{14} = 7/6$, $B_n(1) = B_n$, $n \geq 2$, and

$$B_n'(x) = nB_{n-1}(x), \qquad n \geq 1. \tag{5.13}$$

If we adhere to the definition (5.6) of $S_n(x)$ and integrate by parts, as in (5.9), the left side of (5.12) is found to equal $xS_{13}(x) - S_{14}(x)$.

Example 5. *If* $S_{10}(x)$ *is defined by* (5.7), *then*

$$S_{10}'(x) = 10S_9(x) + \tfrac{5}{66}.$$

Proof. By (5.7) and (5.13),

$$S_{10}'(x) = \frac{B_{11}'(x+1)}{11} = B_{10}(x+1) = 10S_9(x) + B_{10}.$$

Since $B_{10} = \frac{5}{66}$, the proof is complete.

To properly interpret Example 6, we shall need the concept of Bernoulli numbers of fractional index which are defined by Ramanujan in Section 25 of Chapter 5. Taking derivatives in (5.7) and using (5.13), we find that

$$S_n'(x) = nS_{n-1}(x) + B_n, \qquad n \geq 1. \tag{5.14}$$

For the proof of Example 6, we shall define $S_{3/2}'(x)$ by (5.14) with $n = 3/2$ and $B_{3/2}$ replaced by $B_{3/2}^*$. There is a misprint in Example 6 in the notebooks; Ramanujan has written 4 for $2\sqrt{2}$ on the right side of (5.15).

Example 6. *For any real number* x,

$$\int_0^x S_{1/2}(t)\,dt = \tfrac{2}{3}S_{3/2}(x) - \frac{x}{2\pi\sqrt{2}}\zeta\left(\tfrac{3}{2}\right). \tag{5.15}$$

Proof. By (5.14),

$$\begin{aligned}\int_0^x S_{1/2}(t)\,dt &= \frac{2}{3}\int_0^x \left(S_{3/2}'(t) - B_{3/2}^*\right) dt \\ &= \tfrac{2}{3}\left(S_{3/2}(x) - xB_{3/2}^*\right).\end{aligned}$$

From Corollary 1 of Entry 25 in Chapter 5, $B_{3/2}^* = 3\zeta(\frac{3}{2})/(4\pi\sqrt{2})$. This completes the proof.

Entry 6. *Suppose that* φ *is analytic at the origin. If* c_n *denotes the constant of* $\sum f^{(n)}(k)$, *then in some neighborhood of the origin,*

$$\varphi(x) = -\sum_{n=1}^{\infty} \frac{c_n x^n}{n!}.$$

Proof. By the definition of φ given in Entry 3, $\varphi(0)=0$. Extending (5.2), we have

$$\varphi^{(n)}(x)=\sum_{k=1}^{x} f^{(n)}(k)-c_n, \qquad n\geq 1.$$

Hence, $\varphi^{(n)}(0)=-c_n$, $n\geq 1$, and the result follows.

Example 1. *For* $|x|<1$,

$$\operatorname{Log}\Gamma(x+1)=-\gamma x+\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)x^k}{k}. \tag{6.1}$$

The expansion above is well known. (See Henrici's text [2, p. 37].)

Example 2. *For* $|x|<1$,

$$S_{-1}(x)=\sum_{k=1}^{\infty}(-1)^{k+1}\zeta(k+1)x^k. \tag{6.2}$$

With $S_{-1}(x)$ defined by (5.6), Example 2 is obviously false, because the left side of (6.2) is identically 0 for $|x|<1$. However, if we define S_{-1} by (5.11), then we see that (6.2) arises from the differentiation of (6.1).

The complete proof of the following result is given by Ramanujan in the notebooks. For brevity, put

$$\psi(x)=\sum_{k=0}^{n-1}\varphi\left(\frac{x-k}{n}\right) \quad\text{and}\quad \eta(x)=\sum_{k=1}^{x}f\left(\frac{k}{n}\right),$$

where n and x are positive integers.

Entry 7. *Let c'_n be the constant of $\sum f(k/n)$; in particular, $c'_1=c$. Then $\psi(x)-nc=\eta(x)-c'_n$.*

Proof. Since $\psi(k)-\psi(k-1)=f(k/n)$, we find that

$$\psi(x)-\psi(0)=\sum_{k=1}^{x}\{\psi(k)-\psi(k-1)\}=\sum_{k=1}^{x}f\left(\frac{k}{n}\right).$$

Thus, $\psi(x)$ differs from $\eta(x)$ by a constant, namely $\psi(0)$. This constant can be found by determining the constants of $\psi(x)$ and $\eta(x)$, which are, respectively, nc and c'_n. The result follows.

Corollary 1.

$$\sum_{k=1}^{n-1}\varphi\left(\frac{-k}{n}\right)=nc-c'_n.$$

Proof. The left side above is $\psi(0)$, and so the result was obtained in the proof of Entry 7.

The next five corollaries are simple consequences of previous definitions and Corollary 1. There is a misprint in Corollary 2(v) in the notebooks; Ramanujan writes c_4' for c_6'.

Corollary 2(i). $$\varphi(-\tfrac{1}{2}) = 2c - c_2'.$$

Corollary 2(ii). $$c = c_0 = c_1'.$$

Corollary 2(iii). $$\varphi(-\tfrac{1}{3}) + \varphi(-\tfrac{2}{3}) = 3c - c_3'.$$

Corollary 2(iv). $$\varphi(-\tfrac{1}{4}) + \varphi(-\tfrac{3}{4}) = 2c + c_2' - c_4'.$$

Corollary 2(v). $$\varphi(-\tfrac{1}{6}) + \varphi(-\tfrac{5}{6}) = c + c_2' + c_3' - c_6'.$$

Entry 8 is a particular instance of the Euler–Maclaurin summation formula.

The sole purpose of Entry 9 is to assign notation for certain types of infinite series, but the notation is never used in the sequel.

Entry 10(i) constitutes a warning that the constant of a series can only be found from a "regular" series, although Ramanujan does not define a regular series. Different values of c may arise from irregular series, which are termwise equal to regular series.

Entry 10(ii). *In general,*

$$\sum (-1)^{k+1} a_k \begin{cases} \neq \sum (a_{2k-1} - a_{2k}) \\ \neq a_1 - \sum (a_{2k} - a_{2k+1}) \\ = a_1 - \sum (-1)^{k+1} a_{k+1}. \end{cases}$$

Of course, the principles enunciated above are well known. Ramanujan illustrates the two inequalities with the example

$$\sum (-1)^{k+1} k = \tfrac{1}{4}, \qquad \sum \{(2k-1) - 2k\} = \tfrac{1}{2}, \qquad 1 - \sum \{2k - (2k+1)\} = \tfrac{3}{2}.$$

The first equality above was shown in (1.4), while the latter two equalities follow from Example 1 of Entry 1.

Entry 10(iii) merely gives the definition for adding two alternating series.

Example 1.

$$\sum (-1)^{k+1} a_k + \sum (-1)^{k+1} b_k = a_1 + \sum (-1)^{k+1} (b_k - a_{k+1}).$$

Example 1 is a simple consequence of Entry 10(iii) and the equality in Entry 10(ii), which is also used to establish Examples 3 and 4 below. Example 2 follows from Example 1, Example 3 follows from Example 2, and Example 4 follows from Example 3.

Example 2. $$\sum (-1)^{k+1} a_k = \tfrac{1}{2} a_1 + \tfrac{1}{2} \sum (-1)^{k+1} (a_k - a_{k+1}).$$

Example 3.

$$\sum(-1)^{k+1}a_k=\tfrac{1}{4}(3a_1-a_2)+\tfrac{1}{4}\sum(-1)^{k+1}(a_k-2a_{k+1}+a_{k+2}).$$

Example 4.

$$\sum(-1)^{k+1}a_k=\tfrac{1}{8}(7a_1-4a_2+a_3)+\tfrac{1}{8}\sum(-1)^{k+1}(a_k-3a_{k+1}+3a_{k+2}-a_{k+3}).$$

Entry 11. *Put* $\Delta a_k=a_k-a_{k+1}$ *and* $\Delta^n a_1=\Delta(\Delta^{n-1}a_1)$, $n\geq 2$. If $\sum(-1)^{k+1}a_k$ *is convergent, then*

$$\sum(-1)^{k+1}a_k=\sum\frac{\Delta^{k-1}a_1}{2^k}.$$

Entry 11 is known as Euler's transformation of series. (See Knopp's treatise [1, p. 244].) Examples 2–4 serve as motivation for Entry 11.

In Entry 12, Ramanujan claims that if a_2/a_3 lies between a_1/a_2 and a_3/a_4, then $\sum(-1)^{k+1}a_k$ lies between $m_1=a_1^2/(a_1+a_2)$ and $m_2=1-a_2^2/(a_2+a_3)$. Because no restrictions are placed on the terms a_k with $k\geq 5$, it is to be expected that Ramanujan's assertion is false. Even if we assume that a_k/a_{k+1} is monotonically increasing or monotonically decreasing for $k\geq 1$, the proposed theorem is false. For example, define a sequence $\{a_k\}$ by $a_1=5$, $a_2=4$, $a_3=3$, $a_4=1$, and $a_k=(4\cdot 5\cdots(k-1))^{-1}$, $k\geq 5$. It is readily checked that a_k/a_{k+1} is strictly increasing, $m_1=25/9$, $m_2=-9/7$, and $\sum(-1)^{k+1}a_k>3$. Thus, we clearly have established a counterexample to Ramanujan's assertion.

Ramanujan illustrates his purported theorem with three examples. In the first, he considers $\sum(-1)^{k+1}k$ which by (1.4) has the "value" $\frac{1}{4}$. The hypotheses are seen to be satisfied, $m_1=\frac{1}{3}$, and $m_2=\frac{1}{5}$. Thus, the conclusion of the "theorem" is independently verified in this instance.

For his second example, Ramanujan examines $\sum_{k=0}^{\infty}(-1)^k k!$. The hypotheses are easily checked, while $m_1=\frac{1}{2}$ and $m_2=\frac{2}{3}$. Ramanujan claims that the constant for this series is $\frac{3}{5}$ "very nearly." In fact, in his [15, p. 351] first letter to Hardy, Ramanujan gave the value 0.596 ... for the aforementioned series. Moreover, this constant was also calculated by Euler to be approximately 0.5963. (See Bromwich's text [1, p. 324].) Therefore, this example also fits Ramanujan's "theorem."

For a thorough discussion of this example and for some historical references, see Watson's paper [1].

The third example is meaningless because Ramanujan assumes that the converse of his assertion is true. This would be false even if his assertion were true.

We are unable to offer a corrected version of Ramanujan's assertion. However, H. Diamond has kindly pointed out the following result. Let $\{a_k\}$, $1\leq k<\infty$, be a sequence of positive numbers tending to 0. Define

$a_0 = 2a_1 - a_2$ and assume that $a_{k-1}/a_k \geq a_k/a_{k+1}$, $k \geq 2$. Then by a theorem in Katznelson's text [1, p. 22],

$$a_0 + 2\sum_{k=1}^{\infty} a_k \cos(k\theta) \geq 0, \qquad \theta \text{ real}.$$

Putting $\theta = \pi$, we deduce that

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \leq \tfrac{1}{2}(2a_1 - a_2).$$

Section 13 provides a very general discussion on how to accelerate the convergence of a series. Ramanujan then offers several examples in illustration. The first, in fact, is a special case of Euler's transformation of power series. (Consult Bromwich's book [1, p. 62].)

Example (a). *If* $y = x/(1 + x/2)$, $|x| < 1$, *and* $|y| < 2$, *then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = 2 \sum_{k=0}^{\infty} \frac{(y/2)^{2k+1}}{2k+1}.$$

Proof. The left side above is

$$\operatorname{Log}(1+x) = \operatorname{Log}\left(1 + \frac{y}{1 - y/2}\right) = \operatorname{Log}\left(1 + \frac{y}{2}\right) - \operatorname{Log}\left(1 - \frac{y}{2}\right)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(y/2)^k}{k} + \sum_{k=1}^{\infty} \frac{(y/2)^k}{k},$$

and the desired equality follows.

Example (b1). *For* $|x| < \pi^2/4$,

$$\sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k} - 1) B_{2k} x^k}{(2k)!} = \frac{x}{1} + \frac{x}{3} + \frac{x}{5} + \cdots.$$

Example (b1) is due to Lambert and can be found in Wall's book [1, p. 349]. The series on the left side above is, in fact, the Maclaurin series of $\sqrt{x} \tanh \sqrt{x}$. The continued fraction expansion on the right side converges for all x.

Example (b2). *Let x be any complex number which is not nonpositive. Then*

$$\sum_{k=0}^{\infty} \frac{(-1)^k k!}{x^{k+1}} = \frac{1}{x+1} - \frac{1^2}{x+3} - \frac{2^2}{x+5} - \cdots. \tag{13.1}$$

Of course, the infinite series in (13.1) converges for no value of x. However, if we consider

$$I(x) \equiv \int_0^{\infty} \frac{e^{-u}}{x+u}\, du,$$

formally expand $1/(x+u)$ in powers of u, and integrate termwise, we obtain the asymptotic expansion in (13.1). (See Hardy's book [15, pp. 26–29] for a more complete discussion of this.) The continued fraction expansion in (13.1) is a continued fraction expansion of $I(x)$. (See Wall's text [1, p. 356].) Thus, in this sense, Example (b2) is valid.

Example (c). *As x tends to ∞,*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(x+1)(x+2)\cdots(x+k)} \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1}B(k)}{x^k},$$

where $B(k)$ is the kth *Bell number.*

Example (c) is an immediate consequence of Entry 3 in Chapter 3 and the discussion following Example 5(i) of Section 8 in the same chapter.

Example 1. *As x tends to ∞,*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{x+k} \sim \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{(4^k-1)B_{2k}}{2kx^{2k}}.$$

Example 1 is readily obtained by letting $f(t) = 1/(x+t)$ in Boole's summation formula, which can be found in Berndt and Schoenfeld's paper [1, p. 31]. The details are quite straightforward.

Example 2 gives the well-known partial fraction decomposition for $\cot(\pi x)$.

In Example 3, Ramanujan, in essence, attempts to calculate Euler's constant γ by using the fact that

$$\lim_{s\to 1+} \left\{\zeta(s) - \frac{1}{s-1}\right\} = \gamma.$$

More details may be found in Hardy's book [15, pp. 332–333].

In contrast to the formulations in the notebooks, the right sides of (14.1) and (14.4) below are expressed as finite sums with remainders instead of as infinite series. The derivatives of the functions to which we shall apply the Euler–Maclaurin summation formula are not of constant sign on $[0, \infty)$. Hence, the aforementioned general theorem on the remainder in the books of Bromwich [1, pp. 327–329] or Hardy [15, p. 328] does not apply. Thus, in order to make the applications which follow Entries 14(i) and (ii), finite versions with bounds for the remainders are needed. A complete proof of Entry 14(ii) can be found in Schlömilch's book [1, Band 2, pp. 238–241]. Entry 14(i) can be derived from Entry 14(ii).

Entry 14(i). *Let $x > 0$. Then, for $n \geq 1$,*

$$\sum_{k=1}^{\infty} \frac{1}{e^{kx}+1} = \frac{\operatorname{Log} 2}{x} - \frac{1}{4} + \sum_{k=1}^{n} \frac{(2^{2k}-1)B_{2k}^2 x^{2k-1}}{(2k)(2k)!} + R_n, \qquad (14.1)$$

where

$$|R_n| \le \frac{|B_{2n}B_{2n+2}|x^{2n}}{(2n)!}\left\{\frac{x^2}{4\pi^2} + \frac{\pi^2}{6} + 2^{2n+1}\left(\frac{x^2}{\pi^2} + \frac{\pi^2}{6}\right)\right\}. \tag{14.2}$$

Proof. The desired results (14.1) and (14.2) follow from Entry 14(ii) upon realizing that

$$\frac{1}{e^y+1} = \frac{1}{e^y-1} - \frac{2}{e^{2y}-1}.$$

If we use a method of proof similar to that in Schlömilch's book [1, Band 2, pp. 238–241], we can obtain the better estimate

$$|R_n| \le \frac{|B_{2n}B_{2n+2}|(2^{2n+1}-2)x^{2n}}{(2n)!}\left(\frac{x^2}{\pi^2} + \frac{\pi^2}{6}\right) \tag{14.3}$$

for the remainder.

Entry 14(ii). *Let $x > 0$. Then, for $n \ge 1$,*

$$\sum_{k=1}^{\infty} \frac{1}{e^{kx}-1} = \frac{\gamma}{x} - \frac{\operatorname{Log} x}{x} + \frac{1}{4} - \sum_{k=1}^{n} \frac{B_{2k}^2 x^{2k-1}}{(2k)(2k)!} + R_n, \tag{14.4}$$

where γ denotes Euler's constant and where

$$|R_n| \le \frac{|B_{2n}B_{2n+2}|x^{2n}}{(2n)!}\left(\frac{x^2}{4\pi^2} + \frac{\pi^2}{6}\right).$$

Example 1. *The constant for the series* $\sum k^{1/100}$ is $-0.4909\ldots$.

Proof. Put $f(t) = t^{1/100}$ and $a = 0$ in (1.3) and let n tend to ∞. Then

$$C \sim -\frac{100}{101} + \frac{1}{2} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1). \tag{14.5}$$

This is in contrast to what would be obtained from (1.2). A calculation on an HP-35 shows that the first five terms on the right side of (14.5) total $-0.490912753\ldots$. The sixth term is equal to $0.00000580788\ldots$. By a general theorem on the remainder in the Euler–Maclaurin formula (consult the books of Bromwich [1, p. 328] or Hardy [15, p. 328]), the magnitude of this sixth term is greater than the error made by approximating C by the first five terms. The result now follows.

In fact, Ramanujan claims that the constant in Example 1 is equal to -0.4909100 (approximately). This alleged accuracy apparently cannot be obtained from (14.5), because the sixth term appears to be the smallest one in (14.5). Calculations show that the next three terms are successively greater in magnitude.

Example 2. *We have*

$$\sum_{k=1}^{\infty} \frac{1}{2^k+1} \approx \frac{3}{4} + \frac{\operatorname{Log} 2}{48}.$$

Proof. Apply Entry 14(i) with $x = \operatorname{Log} 2$ and $n = 1$. The indicated approximation then arises with an error less than 0.0135617002..., by (14.3).

Example 3. *We have*

$$\sum_{k=1}^{\infty} \frac{1}{(10/9)^k+1} \approx 6.331009.$$

Proof. Apply Entry 14(i) with $x = \operatorname{Log}(10/9)$ and $n = 2$. We then obtain an approximation 6.331008696... with an error less than 0.000000221..., by (14.2).

Example 4. *We have*

$$\sum_{k=1}^{\infty} \frac{1}{(10/9)^k-1} = \text{“27 } \textit{nearly.}\text{”}$$

Proof. Apply Entry 14(ii) with $x = \operatorname{Log}(10/9)$ and $n = 1$. We then get the approximation 27.08648507..., with an error less than 0.00030438.... This justifies the quoted approximation of Ramanujan.

Entry 15(i). *For* $|x| > 1$,

$$\sum_{k=1}^{\infty} \frac{1}{x^k-1} = \sum_{k=1}^{\infty} \frac{x^k+1}{x^{k^2}(x^k-1)}.$$

Entry 15(i) was first stated by Clausen [1] in 1828 and first proved in print by Scherk [1] in 1832.

Proof. We have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{x^k-1} &= \sum_{k=1}^{\infty} \left\{ \sum_{n=1}^{k-1} \frac{1}{x^{kn}} + \frac{1}{x^{k(k-1)}(x^k-1)} \right\} \\ &= \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} x^{-kn} + \sum_{k=1}^{\infty} \frac{1}{x^{k(k-1)}(x^k-1)} \\ &= \sum_{n=1}^{\infty} \frac{1}{x^{n^2}(x^n-1)} + \sum_{k=1}^{\infty} \frac{x^k}{x^{k^2}(x^k-1)},\end{aligned}$$

from which the desired equality follows.

Entry 15(ii). *For* $|x| > 1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{x^k-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x^{2k}+1)}{x^{k^2}(x^{2k}-1)}.$$

Proof. The proof is completely analogous to that of Entry 15(i).

Entry 16. *For each positive integer n,*

$$\sum_{k=1}^{n} \frac{r^k}{1-ax^k} = \sum_{k=1}^{n} \frac{(arx^k)^k}{1-ax^k} + \sum_{k=1}^{n} a^{k-1} \frac{(rx^{k-1})^k - (rx^{k-1})^{n+1}}{1-rx^{k-1}}.$$

Proof. If k is any positive integer, then

$$\sum_{j=1}^{k} r^k a^{j-1} x^{k(j-1)} = \frac{r^k}{1-ax^k} - \frac{(arx^k)^k}{1-ax^k}. \tag{16.1}$$

Summing both sides of (16.1) on k, $1 \le k \le n$, we get

$$\begin{aligned}\sum_{k=1}^{n} \left\{\frac{r^k}{1-ax^k} - \frac{(arx^k)^k}{1-ax^k}\right\} &= \sum_{k=1}^{n} \sum_{j=1}^{k} r^k a^{j-1} x^{k(j-1)} \\ &= \sum_{j=1}^{n} a^{j-1} \sum_{k=j}^{n} r^k x^{(j-1)k}.\end{aligned}$$

Upon summing the inner sum on the far right side above, we complete the proof.

Corollary. *Let* $|r| < 1$ *and* $|x| \le 1$. *If* $|x| = 1$, *we further assume that* $|ar| < 1$. *Then*

$$\sum_{k=1}^{\infty} \frac{r^k}{1-ax^k} = \sum_{k=1}^{\infty} \frac{(arx^k)^k}{1-ax^k} + \sum_{k=1}^{\infty} \frac{a^{k-1} r^k x^{(k-1)k}}{1-rx^{k-1}}.$$

Proof. Let n tend to ∞ in Entry 16.

Entry 17. *Let a and b be arbitrary,* $|x| \le 1$, *and* $|n| < 1$. *If* $|x| = 1$, *assume that* $|mn| < 1$. *Then*

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(a+kb)n^k}{1-mx^k} &= \sum_{k=0}^{\infty} \frac{(a+kb)(1-mnx^{2k})(mnx^k)^k}{(1-mx^k)(1-nx^k)} \\ &\quad + \frac{b}{m} \sum_{k=0}^{\infty} \frac{(mnx^k)^{k+1}}{(1-nx^k)^2}.\end{aligned}$$

Proof. If suffices to show that

$$\sum_{k=0}^{\infty} \frac{n^k}{1-mx^k} = \sum_{k=0}^{\infty} \frac{(1-mnx^{2k})(mnx^k)^k}{(1-mx^k)(1-nx^k)} \tag{17.1}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{kn^k}{1-mx^k} &= \sum_{k=0}^{\infty} \frac{k(1-mnx^{2k})(mnx^k)^k}{(1-mx^k)(1-nx^k)} \\ &\quad + \sum_{k=0}^{\infty} \frac{m^k n^{k+1} x^{k(k+1)}}{(1-nx^k)^2}.\end{aligned} \tag{17.2}$$

In the Corollary of Entry 16 put $a = m$ and $r = n$ to get

$$\sum_{k=0}^{\infty} \frac{n^k}{1 - mx^k} = \sum_{k=0}^{\infty} \frac{(mnx^k)^k}{1 - mx^k} + \sum_{k=0}^{\infty} \frac{m^k n^{k+1} x^{k(k+1)}}{1 - nx^k}. \tag{17.3}$$

By combining the two series on the right side of (17.3) into one series, we readily achieve (17.1).

Differentiating both sides of (17.3) with respect to n, we find that

$$\sum_{k=0}^{\infty} \frac{kn^{k-1}}{1 - mx^k} = \sum_{k=0}^{\infty} \frac{km^k n^{k-1} x^{k^2}}{1 - mx^k} + \sum_{k=0}^{\infty} \frac{km^k n^k x^{k(k+1)}}{1 - nx^k}$$
$$+ \sum_{k=0}^{\infty} \frac{m^k n^k x^{k(k+1)}}{1 - nx^k} + \sum_{k=0}^{\infty} \frac{m^k n^{k+1} x^{k(k+2)}}{(1 - nx^k)^2}.$$

On the right side above, combine the first and second series together and combine the third and fourth series together. Upon multiplying both sides by n, we arrive at (17.2).

We shall not state Corollary 1 which is simply the special case $m = n$ of Entry 17.

Corollary 2. *For* $|x| > 1$,

$$\sum_{n=1}^{\infty} \frac{d(n)}{x^n} = \sum_{k=1}^{\infty} \frac{1}{x^k - 1},$$

where $d(n)$ *denotes the number of positive divisors of* n.

Corollary 2, which follows easily from the identity

$$\sum_{n=1}^{\infty} \frac{d(n)}{x^n} = \sum_{k,n=1}^{\infty} \frac{1}{x^{kn}},$$

is well known and can be found in the classic text of Hardy and Wright [1, Theorem 310, p. 258].

CHAPTER 7

Sums of Powers, Bernoulli Numbers, and the Gamma Function

The principal topics in Chapter 7 concern sums of powers, an extended definition of Bernoulli numbers, the Riemann zeta-function $\zeta(s)$ and allied functions, Ramanujan's theory of divergent series, and the gamma function. This chapter thus represents a continuation of the subject matter of Chapters 5 and 6. Perhaps more so than any other chapter in the second notebook, Chapter 7 offers a considerable amount of numerical calculation. The extent of Ramanujan's calculations is amazing, since he evidently performed them without the aid of a mechanical or electrical device.

In this chapter, we shall frequently state and prove results for complex values of a variable. This is in contrast to Ramanujan who evidently intended his variables to be real. However, to give rigorous proofs, we have frequently needed to use analytic continuation, and so we state theorems in more generality than originally intended. We denote complex variables by s with $\sigma = \text{Re}(s)$, by r with $u = \text{Re}(r)$, and by z.

As might be expected, several of the results in Chapter 7 are not new. For example, Ramanujan rediscovered the functional equation of $\zeta(s)$, found in Entry 4 in somewhat disguised form. As was the case with Euler, Ramanujan had no real proof. It is fascinating how he arrived at this result, with reasoning based on his tenuous theory of the "constant" of a series.

Entry 1. *Let*

$$\varphi_r(x) = \sum_{k=1}^{x} k^r, \tag{1.1}$$

where r is any complex number. Then if $r \neq -1$, as x tends to ∞,

$$\varphi_r(x) \sim \zeta(-r) + \frac{x^{r+1}}{r+1} + \frac{x^r}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}\Gamma(r+1)x^{r-2k+1}}{(2k)!\,\Gamma(r-2k+2)}. \tag{1.2}$$

Proof. Applying the Euler–Maclaurin summation formula (I3) with $f(t) = t^r$, $\alpha = 1$, and $\beta = x$, we find that

$$\varphi_r(x) = c + \frac{x^{r+1}}{r+1} + \frac{x^r}{2} + \sum_{k=1}^{n} \frac{B_{2k}\Gamma(r+1)x^{r-2k+1}}{(2k)!\,\Gamma(r-2k+2)} - \frac{\Gamma(r+1)}{\Gamma(r-2n)} \int_x^\infty P_{2n+1}(t)t^{r-2n-1}\,dt,$$

where

$$c = -\frac{1}{r+1} + \frac{1}{2} - \sum_{k=1}^{n} \frac{B_{2k}\Gamma(r+1)}{(2k)!\,\Gamma(r-2k+2)} + \frac{\Gamma(r+1)}{\Gamma(r-2n)} \int_1^\infty P_{2n+1}(t)t^{r-2n-1}\,dt \tag{1.3}$$

and n is a positive integer with $2n > u$. Note that

$$\int_x^\infty P_{2n+1}(t)t^{r-2n-1}\,dt = O\left(\int_x^\infty t^{u-2n-1}\,dt\right) = O(x^{u-2n}),$$

as x tends to ∞. Thus, it remains to show that

$$c = \zeta(-r). \tag{1.4}$$

From the Euler–Maclaurin formula (I3), we have, for $\sigma > 1$,

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{n} \frac{B_{2k}\Gamma(s+2k-1)}{(2k)!\,\Gamma(s)} - \frac{\Gamma(s+2n+1)}{\Gamma(s)} \int_1^\infty P_{2n+1}(t)t^{-s-2n-1}\,dt. \tag{1.5}$$

By analytic continuation, (1.5) holds for $\sigma + 2n > 0$. Putting $s = -r$ in (1.5) and using (1.3), we obtain (1.4).

The analogue of (1.2) for $r = -1$ is due to Euler and is stated by Ramanujan in Chapter 8, Entry 2.

If r is a nonnegative integer, the series in (1.2) is finite, and we may replace the asymptotic sign by an equality sign. Moreover, (1.2) reduces to the familiar formula

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - B_{r+1}(1)}{r+1}, \tag{1.6}$$

where $B_n(x)$ denotes the nth Bernoulli polynomial.

In Chapter 5, Section 25, Ramanujan defines a Bernoulli number B_r^* of arbitrary index by

$$\zeta(r) = \frac{(2\pi)^r}{2\Gamma(r+1)} B_r^*. \tag{1.7}$$

In particular, if $r = 2n$ is an even positive integer, $B_{2n}^* = (-1)^{n-1} B_{2n}$, and (1.7) reduces to Euler's famous formula for $\zeta(2n)$. Using the functional equation for $\zeta(s)$ (see Entry 4), we find that

$$\zeta(-r) = \frac{B_{r+1}^* \cos\{\pi(r+1)/2\}}{r+1}. \tag{1.8}$$

In Ramanujan's version of (1.2), $\zeta(-r)$ is replaced by the right side of (1.8).

After Entry 1, Ramanujan makes some remarks about the "constant" of a series. This concept was introduced by Ramanujan in Chapter 6. The "constant" in Entry 1 is merely the constant term $\zeta(-r)$ in the asymptotic expansion (1.2).

Now define, for $\sigma > 0$,

$$\eta(s) = \sum_{k=1}^{\infty} (-1)^{k+1} k^{-s}. \tag{2.1}$$

Note that

$$\eta(s) = (1 - 2^{1-s})\zeta(s), \tag{2.2}$$

which, by analytic continuation, is valid for all complex values of s.

Entry 2. *For each complex number r,*

$$\eta(-r) = \frac{(2^{r+1} - 1) B_{r+1}^* \sin(\pi r/2)}{r+1}.$$

Proof. Set $s = -r$ in (2.2) and use (1.8).

We now wish to extend the definition of $\varphi_r(x)$ to encompass complex values of x. First, for $u < 0$, redefine

$$\varphi_r(x) = \sum_{k=1}^{\infty} \{k^r - (k+x)^r\}. \tag{2.3}$$

Observe that, if $u < -1$,

$$\varphi_r(x) = \zeta(-r) - \psi(-r, x+1), \tag{2.4}$$

where $\psi(-r, x+1) = \sum_{k=1}^{\infty} (k+x)^r$. Note that $\psi(s, x)$ is very closely related to the Hurwitz zeta-function $\zeta(s, x)$, except that the latter function is usually defined only for $0 < x \leq 1$. The methods for analytically continuing $\zeta(s, x)$ (see, e.g., Berndt [1], Titchmarsh [3, p. 37], or Whittaker and Watson [1, p. 268]) normally can be easily adapted to establish the analytic continuation of $\psi(s, x)$ as well. Thus, by analytic continuation, we shall now define $\varphi_r(x)$, for all complex values of x and r, by (2.4). Moreover, if $u < 0$ and x is a positive integer, we find from (2.3) that

$$\varphi_r(x) = \sum_{k=1}^{x} k^r. \tag{2.5}$$

By analytic continuation, (2.5) is valid for all complex values of r. Thus, the new definition (2.4) agrees with our former definition (1.1) if x is a positive integer. If r is a nonnegative integer and $-1 < x \le 0$, then by (2.4) and the well-known fact $\psi(-r, x+1) = -B_{r+1}(x+1)/(r+1)$, we find that

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - B_{r+1}(1)}{r+1}.$$

By analytic continuation, this holds for all x, and so we see that (2.4) is in agreement with (1.6) if r is a nonnegative integer and x is complex.

Corollary. *If r is complex and $r \neq -1$, then*

$$\varphi_r(-\tfrac{1}{2}) = \frac{(2-2^{-r})B^*_{r+1}\cos\{\pi(r+1)/2\}}{r+1}.$$

Proof. By (2.3), if $u < 0$, we easily find that

$$\varphi_r(-\tfrac{1}{2}) = -2^{-r}\eta(-r), \tag{2.6}$$

where $\eta(s)$ is defined by (2.1). By analytic continuation, (2.6) is valid for all complex r, $r \neq -1$. Using Entry 2 in (2.6), we complete the proof.

If r is a positive integer, then, by (1.6), the Corollary is equivalent to the well-known fact $B_{r+1}(\tfrac{1}{2}) = -(1-2^{-r})B_{r+1}$ (Abramowitz and Stegun [1, p. 805]).

Entry 3. *Let $\varphi_r(x)$ be defined by* (2.4) *and let a and b be complex numbers with $b \neq 0$. Then*

$$\sum_{k=1}^{x} (a+kb)^r = b^r\left\{\varphi_r\left(x+\frac{a}{b}\right) - \varphi_r\left(\frac{a}{b}\right)\right\}.$$

Proof. For $r < 0$, the desired formula follows easily from (2.3). The result for all r follows by analytic continuation.

Entry 4. *For any complex number r,*

$$\frac{\sin(\pi r/2)B^*_{1-r}}{1-r} = \zeta(r) = \frac{(2\pi)^r}{2\Gamma(r+1)}B^*_r. \tag{4.1}$$

Proof. We present Ramanujan's interesting argument, which is not rigorous.

Rewriting (1.6) in Ramanujan's notation, we have

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - \sin(\pi r/2)B^*_{r+1}}{r+1},$$

where r is a natural number. We now suppose that this formula is valid for *all*

r. The "constant" in this representation for $\varphi_{-r}(x)$ is

$$\frac{\sin(\pi r/2)B^*_{1-r}}{1-r}.$$

On the other hand, from (1.4) and (1.7), the "constant" is also equal to

$$\zeta(r)=\frac{(2\pi)^r B^*_r}{2\Gamma(r+1)}.$$

These constants must be equal, and hence (4.1) follows.

The equalities in (4.1) imply that

$$\zeta(r)=2(2\pi)^{r-1}\Gamma(1-r)\zeta(1-r)\sin\left(\frac{\pi r}{2}\right). \tag{4.2}$$

Mirabile dictu, Ramanujan has derived the functional equation of $\zeta(r)$ (Titchmarsh [3, p. 25]) in a most unorthodox manner!

Corollary 1. *We have* $B^*_{-2}=2\zeta(3)$, $B^*_{-4}=-4\zeta(5)$, $B^*_{-6}=6\zeta(7)$, *and* $B^*_{-8}=-8\zeta(9)$.

Proof. The proposed equalities are special instances of the first equality in (4.1).

Corollary 2. $\Gamma(\frac{1}{2})=\sqrt{\pi}$.

Corollary 3. *For every complex number* z, $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$.

Corollary 2 is the special case $z=\frac{1}{2}$ of the well-known Corollary 3 for which we give Ramanujan's proof.

Proof. Letting $r=-z$ and $r=z+1$ in the extremal sides of (4.1), we find, respectively, that

$$-\frac{\sin(\pi z/2)B^*_{z+1}}{z+1}=\frac{(2\pi)^{-z}B^*_{-z}}{2\Gamma(1-z)}$$

and

$$-\frac{\cos(\pi z/2)B^*_{-z}}{z}=\frac{(2\pi)^{z+1}B^*_{z+1}}{2\Gamma(z+2)}.$$

Multiplying these two equalities together, using the equality $\Gamma(z+2)=(z+1)z\Gamma(z)$, and simplifying, we obtain the desired result.

Corollary 4. *We have*

$$\pi\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k}+\sqrt{2k+2}}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{3/2}}.$$

Ramanujan gives essentially the following faulty proof of Corollary 4. From (2.1) with $s = -\frac{1}{2}$, (2.2), and (4.2), it follows that

$$\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k} + \sqrt{2k+2}} = \frac{\pi}{\sqrt{2}} \sum_{k=0}^{\infty} (-1)^k \{\sqrt{k+1} - \sqrt{k}\}$$
$$= \pi\sqrt{2} \sum_{k=1}^{\infty} (-1)^{k+1}\sqrt{k}$$
$$= \pi\sqrt{2}(1 - 2^{3/2})\zeta(-\tfrac{1}{2})$$
$$= (1 - 2^{-3/2})\zeta(\tfrac{3}{2})$$
$$= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}}.$$

Corollary 4, in fact, is the special case $p = \frac{1}{2}$ of the identity

$$\pi^{p+1} \sum_{k=0}^{\infty} (-1)^k \{(k+1)^p - k^p\} = 4 \sin\left(\frac{\pi p}{2}\right) \Gamma(p+1) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{p+1}},$$

proved in Titchmarsh's text [1, p. 154], where $0 < p < 1$.

Ramanujan [11], [15, pp. 47–49] has established some other results which are akin to Corollary 4. Kesava Menon [1] has given simpler proofs of Ramanujan's results and has proved additional results of this type as well.

Corollary 5. *Let $\eta(s)$ be defined by* (2.1). *Then*

$$(2\pi)^{2/3}\eta(\tfrac{1}{3}) = (1 + 2^{1/3})\Gamma(\tfrac{2}{3})\eta(\tfrac{2}{3}).$$

Proof. Using (2.2), we rewrite the functional equation (4.2) for $\zeta(r)$ in terms of $\eta(r)$ to get

$$(1 - 2^r)\eta(r) = 2(2\pi)^{r-1}(1 - 2^{1-r}) \sin\left(\frac{\pi r}{2}\right) \Gamma(1-r)\eta(1-r).$$

Putting $r = \frac{1}{3}$ and simplifying, we achieve the desired result.

Corollary 6. *As x tends to ∞,*

$$\sum_{k=1}^{x} \frac{1}{\sqrt{k}} \sim 2\sqrt{x} + \zeta(\tfrac{1}{2}) + \frac{1}{2\sqrt{x}} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1 \cdot 3 \cdots (4k-3)}{2^{2k-1}} x^{-2k+1/2}. \quad (4.3)$$

This asymptotic formula is a special case of (1.2) but is different from that claimed by Ramanujan (p. 79), who asserts that, as x tends to ∞,

$$\sum_{k=1}^{x} \frac{1}{\sqrt{k}} \sim \sqrt{2 + 4x} + \zeta(\tfrac{1}{2}). \quad (4.4)$$

Formulas (4.3) and (4.4) are incompatible since

$$\sqrt{2 + 4x} = 2\sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{16x^{3/2}} + \cdots;$$

the leading two terms above agree with (4.3), but the third term does not coincide with the corresponding term $-x^{-3/2}/24$ in (4.3). Ramanujan gives no indication as to how he arrived at the approximation $\sqrt{2+4x}$.

Corollary 7. *As x tends to* ∞,

$$\sum_{k=1}^{x} \sqrt{k} \sim \tfrac{2}{3}x^{3/2} + \tfrac{1}{2}x^{1/2} - \frac{1}{4\pi}\zeta(\tfrac{3}{2}) + \tfrac{1}{24}x^{-1/2} + \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1 \cdot 3 \cdots (4k-5)}{2^{2k-1}} x^{-2k+3/2}. \tag{4.5}$$

Using the functional equation (4.2), we observe that Corollary 7 is a special instance of Entry 1, but (4.5) is not the result claimed by Ramanujan. Instead, he has proposed that

$$\sum_{k=1}^{x} \sqrt{k} \sim \tfrac{2}{3}\sqrt{(x+\tfrac{1}{4})(x+\tfrac{1}{2})(x+\tfrac{3}{4})} - \frac{1}{4\pi}\zeta(\tfrac{3}{2}), \tag{4.6}$$

as x tends to ∞. Since the infinite series in (4.5) diverges while

$$\tfrac{2}{3}\sqrt{(x+\tfrac{1}{4})(x+\tfrac{1}{2})(x+\tfrac{3}{4})} = \tfrac{2}{3}x^{3/2} + \tfrac{1}{2}x^{1/2} + \tfrac{1}{24}x^{-1/2} + \cdots$$

converges in a neighborhood of $x = \infty$, (4.5) and (4.6) are certainly not compatible. However, note that the right side of (4.6) does provide a good approximation for the left side.

A similar type of approximation for $\varphi_{1/2}(x)$ has been obtained by Gates [1].

Corollary 8. *As x tends to* ∞,

$$\sum_{k=1}^{x} k^{3/2} \sim \tfrac{2}{5}x^{5/2} + \tfrac{1}{2}x^{3/2} + \tfrac{1}{8}x^{1/2} - \frac{3}{16\pi^2}\zeta(\tfrac{5}{2}) - 3\sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1 \cdot 3 \cdots (4k-7)}{2^{2k-1}} x^{-2k+5/2}. \tag{4.7}$$

Corollary 8 follows from Entry 1 and the functional equation (4.2). In contrast to (4.7), Ramanujan claims that

$$\sum_{k=1}^{x} k^{3/2} \sim \tfrac{2}{5}(x(x+\tfrac{1}{4})(x+\tfrac{1}{2})(x+\tfrac{3}{4})(x+1) + \tfrac{5}{768}(x+\tfrac{1}{2}))^{1/2} - \frac{3}{16\pi^2}\zeta(\tfrac{5}{2}), \tag{4.8}$$

as x tends to ∞. By the same reasoning used in conjunction with Corollary 7, (4.7) and (4.8) are incompatible. However,

$$\tfrac{2}{5}(x(x+\tfrac{1}{4})(x+\tfrac{1}{2})(x+\tfrac{3}{4})(x+1)+\tfrac{5}{768}(x+\tfrac{1}{2}))^{1/2}$$
$$=\tfrac{2}{5}x^{5/2}+\tfrac{1}{2}x^{3/2}+\tfrac{1}{8}x^{1/2}+\ldots.$$

Entry 5. *Let a and b be complex numbers with $b \neq 0$ and a/b not a negative integer. Then if $u < -1$,*

$$\sum_{k=1}^{\infty}(-1)^{k+1}(a+kb)^r=(2b)^r\left\{\varphi_r\left(\frac{a}{2b}\right)-\varphi_r\left(\frac{a-b}{2b}\right)\right\}. \tag{5.1}$$

Proof. We have

$$\sum_{k=1}^{\infty}(-1)^{k+1}(a+kb)^r=\sum_{k=1}^{\infty}(a+(2k-1)b)^r-\sum_{k=1}^{\infty}(a+2kb)^r,$$

and the desired equality follows immediately from the definition (2.3).

There is a misprint in the notebooks, p. 79; Ramanujan has written b^r instead of $(2b)^r$ on the right side of (5.1).

Entry 6(i). *Let x be a positive integer and assume that $n > 0$. Then*

$$(x^2+x)^n=2\sum_{k=0}^{\infty}\binom{n}{2k+1}\varphi_{2n-2k-1}(x).$$

Proof. The proof is indicated by Ramanujan. Expanding by the binomial theorem, we have, for $|z| \geq 1$ and $n > 0$,

$$(z^2+z)^n-(z^2-z)^n=z^{2n}\left\{\left(1+\frac{1}{z}\right)^n-\left(1-\frac{1}{z}\right)^n\right\}$$
$$=z^{2n}\left\{\sum_{k=0}^{\infty}\binom{n}{k}z^{-k}-\sum_{k=0}^{\infty}\binom{n}{k}(-z)^{-k}\right\}$$
$$=2z^{2n}\sum_{k=0}^{\infty}\binom{n}{2k+1}z^{-2k-1}. \tag{6.1}$$

Now set $z=j$ in (6.1) and sum both sides on j, $1 \leq j \leq x$, to get

$$(x^2+x)^n=2\sum_{j=1}^{x}j^{2n}\sum_{k=0}^{\infty}\binom{n}{2k+1}j^{-2k-1}.$$

The required formula follows upon inverting the order of summation and employing (1.1).

Entry 6(ii). *Under the same hypotheses as Entry 6(i), we have*

$$(x+\tfrac{1}{2})(x^2+x)^n=\sum_{k=0}^{\infty}\left\{2\binom{n}{2k+1}+\binom{n}{2k}\right\}\varphi_{2n-2k}(x).$$

Proof. Proceeding in the same fashion as in the previous proof, we find that, for $|z| \geq 1$ and $n > 0$,

$$(z+\tfrac{1}{2})(z^2+z)^n - (z-\tfrac{1}{2})(z^2-z)^n = z^{2n} \sum_{k=0}^{\infty} \left\{ 2\binom{n}{2k+1} + \binom{n}{2k} \right\} z^{-2k}. \tag{6.2}$$

Letting $z = j$ in (6.2) and summing both sides on j, $1 \leq j \leq x$, we arrive at the formula that we sought with no difficulty.

Note that if n is a positive integer, then Entries 6(i) and 6(ii) are valid for all x because they yield polynomial identities.

Corollary 1. *Let* $y = x^2 + x$ *and* $a = x + \frac{1}{2}$. *Then*

$$\varphi_1(x) = \tfrac{1}{2}y, \qquad \varphi_2(x) = \tfrac{1}{3}ay, \qquad \varphi_3(x) = \tfrac{1}{4}y^2,$$
$$\varphi_4(x) = \tfrac{1}{5}ay(y - \tfrac{1}{3}), \qquad \varphi_5(x) = \tfrac{1}{6}y^2(y - \tfrac{1}{2}),$$
$$\varphi_6(x) = \tfrac{1}{7}ay(y^2 - y + \tfrac{1}{3}), \qquad \varphi_7(x) = \tfrac{1}{8}y^2(y^2 - \tfrac{4}{3}y + \tfrac{2}{3}),$$
$$\varphi_8(x) = \tfrac{1}{9}ay(y^3 - 2y^2 + \tfrac{9}{5}y - \tfrac{3}{5}),$$
$$\varphi_9(x) = \tfrac{1}{10}y^2(y-1)(y^2 - \tfrac{3}{2}y + \tfrac{3}{2}),$$
$$\varphi_{10}(x) = \tfrac{1}{11}ay(y-1)(y^3 - \tfrac{7}{3}y^2 + \tfrac{10}{3}y - \tfrac{5}{3}),$$

and

$$\varphi_{11}(x) = \tfrac{1}{12}y^2(y^4 - 4y^3 + \tfrac{17}{2}y^2 - 10y + 5).$$

Proof. The proposed odd indexed formulas for $\varphi_r(x)$ follow from Entry 6(i) by successively letting $n = 1, 2, \ldots, 6$. The proposed even indexed formulas arise from Entry 6(ii) by successively setting $n = 1, 2, \ldots, 5$.

Although the formulas in Corollary 1 have long been known and are instances of (1.6), Ramanujan's method for determining them by means of Entries 6(i) and (ii) is particularly brief and elegant. For other formulas and methods for finding $\varphi_r(x)$ when r and x are positive integers, see a survey paper by D. R. Snow [1] which contains several references.

Corollary 2. *For each positive integer* n, *we have*

(i) $$\sum_{k=1}^{n} \left(\frac{2k-1+\sqrt{5}}{9}\right)^9 = \varphi_9\left(\frac{2n-1+\sqrt{5}}{2}\right)$$

and

(ii) $$\sum_{k=1}^{n} \left(\frac{2k-1+\sqrt{5}}{2}\right)^{10} = \varphi_{10}\left(\frac{2n-1+\sqrt{5}}{2}\right).$$

If p and n are positive integers with n even, then

(iii)
$$\sum_{k=1}^{p} (2k-1)^n = 2^n \varphi_n(p-\tfrac{1}{2}).$$

Proof. Applying Entry 3 with $r=9$, $x=n$, $a=(\sqrt{5}-1)/2$, and $b=1$, we find that

$$\sum_{k=1}^{n} \left(\frac{2k-1+\sqrt{5}}{2}\right)^9 = \varphi_9\left(\frac{2n-1+\sqrt{5}}{2}\right) - \varphi_9\left(\frac{\sqrt{5}-1}{2}\right).$$

However, by Corollary 1, it is easily seen that $(\sqrt{5}-1)/2$ is a root of $\varphi_9(x)$. Hence, part (i) follows.

Part (ii) follows in the same fashion as part (i), except now we use the fact that $\varphi_{10}((\sqrt{5}-1)/2)=0$.

To prove (iii), apply Entry 3 with $r=n$, $x=p$, $a=-1$, and $b=2$ to get

$$\sum_{k=1}^{p} (2k-1)^n = 2^n\{\varphi_n(p-\tfrac{1}{2}) - \varphi_n(-\tfrac{1}{2})\}.$$

By the Corollary to Entry 2, $\varphi_n(-\frac{1}{2})=0$, and the proof is complete.

Entry 7. *If r is a positive integer, then*

$$\varphi_r(x-1) + (-1)^r \varphi_r(-x) = 0.$$

Proof. By (1.6),

$$\varphi_r(x-1) + (-1)^r \varphi_r(-x) = \frac{B_{r+1}(x) + (-1)^r B_{r+1}(1-x)}{r+1}.$$

By a very familiar property of Bernoulli polynomials (Abramowitz and Stegun [1, p. 804]), the right-hand side above is equal to 0.

Corollary. *If r is a positive integer exceeding 1, then $\varphi_r(x)$ is divisible by $x^2(x+1)^2$ or $x(x+\frac{1}{2})(x+1)$ according as r is odd or even.*

Proof. This result follows easily from Entries 6(i) and (ii) by induction on r.

Entry 8. *If r is a positive integer, then*

$$\varphi_r(x) = \frac{1}{r+1}\sum_{k=0}^{r}\binom{r+1}{k} B_k x^{r+1-k} + x^r$$

$$= \frac{x^{r+1}}{r+1} + \frac{x^r}{2} - 2\sum_{k=1}^{[r/2]}\binom{r+1}{2k} k\zeta(1-2k)x^{r+1-2k}. \tag{8.1}$$

Proof. Using the well-known formula (Abramowitz and Stegun [1, p. 804])

$$B_r(x) = \sum_{k=0}^{r} \binom{r}{k} B_k x^{r-k}, \qquad r \geq 0,$$

in (1.6), we find that

$$\varphi_r(x) = \varphi_r(x-1) + x^r = \frac{1}{r+1} \sum_{k=0}^{r+1} \binom{r+1}{k} B_k x^{r+1-k} - \frac{B_{r+1}}{r+1} + x^r,$$

from which the first equality of (8.1) readily follows.

The latter equality of (8.1) follows from (4.1).

Entry 9 is simply a restatement of (2.3).

Entry 10. *For each complex number r and each positive integer n,*

$$\varphi_r(x) - n^r \sum_{k=0}^{n-1} \varphi_r\left(\frac{x-k}{n}\right) = (1 - n^{r+1})\zeta(-r)$$
$$= (n^{r+1} - 1)\frac{\sin(\pi r/2) B_{r+1}^*}{r+1}. \tag{10.1}$$

Proof. For $u < -1$, (2.3) yields

$$\varphi_r(x) - n^r \sum_{k=0}^{n-1} \varphi_r\left(\frac{x-k}{n}\right)$$
$$= \zeta(-r) - \sum_{j=1}^{\infty} (j+x)^r - n^{r+1}\zeta(-r) + \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (nj - k + x)^r$$
$$= (1 - n^{r+1})\zeta(-r). \tag{10.2}$$

By analytic continuation, the extremal sides of (10.2) are equal for all r. The second equality in (10.1) follows from (4.1).

Corollary. *Under the hypotheses of Entry* 10,

$$\sum_{k=1}^{n-1} \varphi_r(-k/n) = (n - n^{-r})\zeta(-r).$$

Proof. Put $x = 0$ in Entry 10 and use the fact that $\varphi_r(0) = 0$ for each r.

Entry 11. *If r is a positive integer, then*

$$\varphi_{-r}(x-1)+(-1)^r\varphi_{-r}(-x)$$
$$=\{1+(-1)^r\}\zeta(r)+\frac{(-1)^r}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}(\pi\cot(\pi x)), \tag{11.1}$$

where if $r=1$, *the first expression on the right side of* (11.1) *is understood to be equal to* 0.

Proof. By (2.3),

$$\varphi_{-r}(x-1)+(-1)^r\varphi_{-r}(-x)$$
$$=\{1+(-1)^r\}\zeta(r)-\sum_{k=0}^{\infty}\left\{\frac{1}{(k+x)^r}+\frac{1}{(x-k-1)^r}\right\}.$$

Since

$$\pi\cot(\pi x)=\sum_{k=0}^{\infty}\left\{\frac{1}{k+x}+\frac{1}{x-k-1}\right\},$$

equality (11.1) now easily follows.

In the notebooks, p. 81, Ramanujan gives (11.1) with r replaced by $-r$ and states that the result is obtained by differentiating the equality $\varphi_{-1}(x-1)-\varphi_{-1}(-x)=-\pi\cot(\pi x)$ a total of r times. The correct number of differentiations is $-r-1$, however. Ramanujan then indicates that (11.1) holds for negative as well as positive values of r and that Entry 7 can thus be deduced. If we interpret

$$\frac{(-1)^r}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}\cot(\pi x)$$

as being identically 0 for $r<0$, then, indeed, we obtain Entry 7 (but this does *not prove* Entry 7).

Ramanujan next indicates a method for calculating the derivatives of $\cot(\pi x)$. We are not certain what Ramanujan's method is, but it seems to be a more complicated version of the simple method which we describe below. This method has been judiciously applied and generalized by Carlitz and Scoville [1], [2]. Set $y=\cot(\pi x)$. Then $\tan^{-1}(1/y)=\pi x$. Upon differentiating both sides of the latter equality with respect to x, we find that

$$\frac{dy}{dx}=-\pi(y^2+1). \tag{11.2}$$

Further derivatives of $\cot(\pi x)$ can be found by successively differentiating (11.2). In this manner, the following table of derivatives of $\cot(\pi x)$ may rapidly be calculated. All formulas are correctly given by Ramanujan, except that he has written 2385 for 2835 in the last denominator of the last entry.

k	$\dfrac{(-1)^k}{\pi^k k!}\dfrac{dy^k}{dx^k}$
0	y
1	y^2+1
2	y^3+y
3	$y^4+\frac{4}{3}y^2+\frac{1}{3}$
4	$y^5+\frac{5}{3}y^3+\frac{2}{3}y$
5	$y^6+2y^4+\frac{17}{15}y^2+\frac{2}{15}$
6	$y^7+\frac{7}{3}y^5+\frac{77}{45}y^3+\frac{17}{45}y$
7	$y^8+\frac{8}{3}y^6+\frac{12}{5}y^4+\frac{248}{315}y^2+\frac{17}{315}$
8	$y^9+3y^7+\frac{16}{5}y^5+\frac{88}{63}y^3+\frac{62}{315}y$
9	$y^{10}+\frac{10}{3}y^8+\frac{37}{9}y^6+\frac{424}{189}y^4+\frac{1382}{2835}y^2+\frac{62}{2835}$

Corollary. *If r is any complex number, then*

(i) $$\varphi_r(x)-2^r\left\{\varphi_r\left(\frac{x}{2}\right)+\varphi_r\left(\frac{x-1}{2}\right)\right\}=(1-2^{r+1})\zeta(-r),$$

(ii) $$\varphi_r(-\tfrac{1}{2})=(2-2^{-r})\zeta(-r),$$

(iii) $$\varphi_r(-\tfrac{1}{3})+\varphi_r(-\tfrac{2}{3})=(3-3^{-r})\zeta(-r),$$

(iv) $$\varphi_r(-\tfrac{1}{4})+\varphi_r(-\tfrac{3}{4})=(2+2^{-r}-4^{-r})\zeta(-r),$$

and

(v) $$\varphi_r(-\tfrac{1}{6})+\varphi_r(-\tfrac{5}{6})=(1+2^{-r}+3^{-r}-6^{-r})\zeta(-r).$$

Proof. Part (i) is the case $n=2$ of Entry 10. Parts (ii)–(v) follow from the Corollary to Entry 10 by successively setting $n=2$, 3, 4, and 6, respectively.

Examples. *If r is a positive, odd integer, then*

(i) $$\varphi_r(-\tfrac{1}{3})=(3-3^{-r})\zeta(-r)/2,$$

(ii) $$\varphi_r(-\tfrac{1}{4})=(1+2^{-r-1}-2^{-2r-1})\zeta(-r),$$

(iii) $$\varphi_r(-\tfrac{1}{6})=(1+2^{-r}+3^{-r}-6^{-r})\zeta(-r)/2,$$

(iv) $$\varphi_r(-\tfrac{1}{5})+\varphi_r(-\tfrac{2}{5})=(5-5^{-r})\zeta(-r)/2,$$

(v) $$\varphi_r(-\tfrac{1}{8})+\varphi_r(-\tfrac{3}{8})=(2+2^{-2r-1}-2^{-3r-1})\zeta(-r),$$

(vi) $\varphi_r(-\frac{1}{10}) + \varphi_r(-\frac{3}{10}) = (3 + 2^{-r} + 5^{-r} - 10^{-r})\zeta(-r)/2,$

and

(vii) $\varphi_r(-\frac{1}{12}) + \varphi_r(-\frac{5}{12}) = (4 - 2^{-r} + 4^{-r} + 6^{-r} - 12^{-r})\zeta(-r)/2.$

Proof. All of these formulas are easily established with the use of the Corollary of Entry 10 and Entry 7. For illustration, we shall give the proof of part (vii). By the aforementioned results,

$$(12 - 12^{-r})\zeta(-r) = \sum_{k=1}^{11} \varphi_r\left(\frac{-k}{12}\right) = 2\sum_{k=1}^{5} \varphi_r\left(\frac{-k}{12}\right) + \varphi_r(-\tfrac{1}{2}).$$

Using Examples (i), (ii), and (iii) and Corollary (ii), we find that

$$\begin{aligned}\varphi_r(-\tfrac{1}{12}) + \varphi_r(-\tfrac{5}{12}) = (12 - 12^{-r})\zeta(-r)/2 \\ - (3 - 3^{-r})\zeta(-r)/2 - (1 + 2^{-r-1} - 2^{-2r-1})\zeta(-r) \\ - (1 + 2^{-r} + 3^{-r} - 6^{-r})\zeta(-r)/2 - (2 - 2^{-r})\zeta(-r)/2,\end{aligned}$$

which, upon simplification, yields (vii).

Ramanujan has incorrectly given the right sides of (vi) and (vii) (p. 83). The examples above are more commonly expressed in terms of values of Bernoulli polynomials. For example, see the handbook of Abramowitz and Stegun [1, pp. 805, 806].

Entry 12. *For every complex number r,*

$$2^r\{\varphi_r(-\tfrac{1}{6}) - \varphi_r(-\tfrac{5}{6})\} = (2^r + 1)\{\varphi_r(-\tfrac{1}{3}) - \varphi_r(-\tfrac{2}{3})\}. \tag{12.1}$$

Proof. Putting $n = 2$ and $x = -\frac{1}{3}$ and $x = -\frac{2}{3}$ in Entry 10, we find that, respectively,

$$\varphi_r(-\tfrac{1}{3}) - 2^r\{\varphi_r(-\tfrac{1}{6}) + \varphi_r(-\tfrac{2}{3})\} = (1 - 2^{r+1})\zeta(-r) \tag{12.2}$$

and

$$\varphi_r(-\tfrac{2}{3}) - 2^r\{\varphi_r(-\tfrac{1}{3}) + \varphi_r(-\tfrac{5}{6})\} = (1 - 2^{r+1})\zeta(-r). \tag{12.3}$$

Subtracting (12.3) from (12.2) and rearranging terms, we deduce (12.1).

The proof above is given by Ramanujan in the notebooks, but he has inadvertently multiplied the right sides of (12.2) and (12.3) by -1.

Example 1. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \tfrac{7}{8}\zeta(3).$$

Example 2. $$\sum_{k=0}^{\infty} \frac{1}{(3k+1)^3} = \frac{2\pi^3}{81\sqrt{3}} + \frac{13}{27}\zeta(3).$$

Example 3. $$\sum_{k=0}^{\infty} \frac{1}{(4k+1)^3} = \frac{\pi^3}{64} + \frac{7}{16}\zeta(3).$$

Example 4. $$\sum_{k=0}^{\infty} \frac{1}{(6k+1)^3} = \frac{\pi^3}{36\sqrt{3}} + \frac{91}{216}\zeta(3).$$

All of these examples follow from well-known general formulas. Example 1 is trivial. Examples 2–4 follow from general formulas for $\varphi_{-2n-1}(-\frac{2}{3})$, $\varphi_{-2n-1}(-\frac{3}{4})$, and $\varphi_{-2n-1}(-\frac{5}{6})$ that can be found in Hansen's tables [1, formulas (6.3.10), (6.3.18), and (6.3.23), pp. 118, 119].

Entry 13. *For each nonnegative integer k, define*

$$c_k = \lim_{m\to\infty} \left(\sum_{j=1}^{m} \frac{\operatorname{Log}^k j}{j} - \frac{\operatorname{Log}^{k+1} m}{k+1} \right). \tag{13.1}$$

Then for all s,

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k c_k}{k!}(s-1)^k. \tag{13.2}$$

In particular, if $A_k = (-1)^k c_k/k!$, $0 \le k < \infty$, *then*

$$A_0 = \gamma = 0.5772156649,$$

$$A_1 = 0.0728158455,$$

$$A_2 = -0.00485,$$

and

$$A_3 = -0.00034, \tag{13.3}$$

where γ *denotes Euler's constant.*

Ramanujan did not explicitly define c_k by (13.1). Instead, he says that c_k is the constant of $\sum_{j=1}^{\infty} (\operatorname{Log}^k j)/j$, but this is equivalent to (13.1). The values of A_k, $0 \le k \le 3$, are correct to the given number of decimal places.

The Laurent series (13.2) has been independently discovered several times in the literature. Apparently, Stieltjes [2] first established (13.2) in 1885. Furthermore, Stieltjes [2, letters 73, 74, 75, 77] and Hermite have thoroughly discussed this result in an exchange of letters. Not surprisingly, the constants A_k are now called Stieltjes constants. In 1887, J. L. W. V. Jensen [1] rediscovered (13.2). Hardy [7], [18, pp. 475–476] and Ramanujan [14], [15, p. 134] himself each stated (13.2) without proof. Briggs and Chowla [1] rediscovered (13.2) again in 1955. Later proofs have been given by Verma [1] and Ferguson [1] in 1963 and Lammel [1] in 1966. Kluyver [1] has established an infinite series representation for c_k. Zhang [1] rediscovered (13.2) and obtained other formulas involving c_k.

Wilton [1], Berndt [1], and Lavrik [1] have evaluated the Laurent coefficients of the Hurwitz zeta-function. Further generalizations to other

Dirichlet series have been found by Briggs and Buschman [1], Knopfmacher [1], and Balakrishnan [2].

Numerical calculations of the constants c_k were first carried out by Jensen [1] who calculated the first 9 coefficients to 9 decimal places. In 1895, Gram [1] published a table of the first 16 coefficients to 16 decimal places. The most extensive calculations to date have been by Liang and Todd [1] who calculated the first 20 coefficients to 15 decimal places.

Briggs [1] and Mitrović [1] have proved theorems on the signs of the coefficients c_k. Uniform bounds for $|c_k|$ have been established by Briggs [1]. Improvements were later made by Berndt [1], Israilov [1], and Balakrishnan [1].

Example 1. *For $|n|$ sufficiently small, we have*

$$\zeta(1+n)+\zeta(1-n)=\frac{2\gamma}{1+0.00839n^2-0.0001n^4+\cdots}.$$

Proof. From Entry 13, for $|n|$ sufficiently small,

$$\begin{aligned}\zeta(1+n)+\zeta(1-n)&=2\gamma+2A_2n^2+2A_4n^4+\cdots\\&=\frac{2\gamma}{1-\dfrac{A_2}{\gamma}n^2+\left(\dfrac{A_2^2}{\gamma^2}-\dfrac{A_4}{\gamma}\right)n^4+\cdots}.\end{aligned}$$

Using the values of $A_0(\gamma)$, A_2, and A_4 given by Liang and Todd [1] and employing a calculator, we complete the proof.

In Example 1, Ramanujan, in fact, has written $+0.0001n^4$ instead of $-0.0001n^4$. Several of the following examples also need corrections.

Example 2. $\zeta(\frac{11}{10})=10.58444842.$

Example 3. $\zeta(\frac{3}{2})=2.6123752.$

Example 4. $\zeta(\frac{5}{2})=1.341490.$

Example 5. $B^*_{3/2}=0.4409932;\quad B^*_{1/2}=-1.032627.$

Example 6. $B^*_{1/3}=-0.9420745;\quad B^*_{-1/3}=-1.3841347.$

Example 7. $B^*_{-1/2}=-1.847228.$

According to Gram's [2] table of values for $\zeta(s)$ which has been reproduced in Dwight's tables [1], the last recorded digit for $\zeta(\frac{11}{10})$ should be 6 rather than 2. These same tables indicate that the last recorded digit for $\zeta(\frac{3}{2})$ is

3 and not 2 and that the last two digits of $\zeta(\frac{5}{2})$ are 87 instead of 90. In an earlier table of Glaisher [1], the values of $\zeta(\frac{11}{10})$ and $\zeta(\frac{3}{2})$ are found to six decimal places.

The five particular values of B_r^* given by Ramanujan can be found by employing (4.1) in conjunction with tabulated values of the Riemann zeta-function. Using the value of $\zeta(\frac{3}{2})$, we find that the last digit of $B_{3/2}^*$ should be 3 rather than 2. The given values of $B_{1/2}^*$ and $B_{-1/2}^*$ are correct. To calculate $B_{1/3}^*$ and $B_{-1/3}^*$ we need the values of $\zeta(\frac{2}{3})$ and $\zeta(\frac{4}{3})$ which are not found in the aforementioned tables but which have been calculated by Hansen and Patrick [1]. Accordingly, the last digit of $B_{1/3}^*$ should be 3 rather than 5. Ramanujan's value of $B_{-1/3}^*$, in contrast to his other calculations, is somewhat off from the correct value -1.3860016.

For a list of all tables of the Riemann zeta-function before 1962, consult the Index of Fletcher, Miller, Rosenhead, and Comrie [1]. The most extensive computations of $\zeta(s)$ appear to have been done by McLellan in 1968; see Wrench's review [1] for a description of these tables.

Entry 14. *Let $n > 0$. Then as n tends to 0,*

$$\sum_{k=2}^{\infty} \frac{1}{k(k^n - 1)} \sim \frac{a_0 - \operatorname{Log} n}{n} + a_1 + \sum_{k=1}^{\infty} a_{k+1} n^{2k-1}, \tag{14.1}$$

where

$$a_0 = \lim_{m \to \infty} \left(\sum_{k=2}^{m} \frac{1}{k \operatorname{Log} k} - \operatorname{Log} \operatorname{Log} m \right) = 0.7946786, \tag{14.2}$$

$$a_1 = \tfrac{1}{2}(1 - \gamma) = 0.2113922,$$

and

$$a_{k+1} = -\frac{B_{2k} A_{2k-1}}{2k}, \qquad k \geq 1,$$

where B_j denotes the jth *Bernoulli number and A_j is defined in Entry* 13, $0 \leq j < \infty$. *In particular,* $a_2 = -0.0060680$ *and* $a_3 = -0.000000475$.

The numerical value for $a_0 + \operatorname{Log} \operatorname{Log} 2$ is found in an article of Boas [2, p. 156]. Boas records the first six digits of $a_0 + \operatorname{Log} \operatorname{Log} 2$ in [1, p. 244]. The numerical values for a_1, a_2, and a_3 may be determined from (13.3), or, more accurately, from the table of Liang and Todd [1]. While Ramanujan correctly gives a_1 and a_2, his value -0.0000028 for a_3 is incorrect.

Proof. Let $t \geq 1$, $x \geq 0$, and suppose that $0 < n < A$, where A is fixed and positive but otherwise arbitrary. Define

$$f(t) = \frac{1}{t(t^n - 1)}, \qquad h(t) = f(t) - \frac{1}{nt \operatorname{Log} t}, \quad \text{and} \quad g(x) = \frac{1}{e^x - 1} - \frac{1}{x}.$$

Then $h(t) = g(n \operatorname{Log} t)/t$ and

$$h'(t) = t^{-2}\{ng'(n \operatorname{Log} t) - g(n \operatorname{Log} t)\}. \tag{14.3}$$

Fix an integer $N \geq 1$. Applying Taylor's theorem to g and to g', we see from (14.3) that

$$t^2 h'(t) = \sum_{j=0}^{N} \frac{(n \operatorname{Log} t)^j}{j!} \{ng^{(j+1)}(0) - g^{(j)}(0)\}$$
$$+ \frac{(n \operatorname{Log} t)^{N+1}}{(N+1)!} \{ng^{(N+2)}(\theta_1) - g^{(N+1)}(\theta_2)\}, \qquad (14.4)$$

where $0 \leq \theta_1, \theta_2 \leq n \operatorname{Log} t$. By the definition (I1) for the Bernoulli numbers,

$$\frac{g^{(j)}(0)}{j!} = \frac{B_{j+1}}{(j+1)!}, \qquad j \geq 0. \qquad (14.5)$$

Thus, as x tends to 0, $g^{(N)}(x)$ tends to $B_{N+1}/(N+1)$. Using the fact that

$$g(x) = -\frac{1}{x} + \sum_{k=0}^{\infty} e^{-(k+1)x}, \qquad x > 0,$$

we find that $g^{(N)}(x)$ tends to 0 as x tends to ∞. Hence, $g^{(N)}(x)$ is bounded for each fixed N, and so the last expression in (14.4) is $O(\{n \operatorname{Log} t\}^{N+1})$, where the implied constant is independent of n and t. Using (14.5), we deduce from (14.4) that

$$t^2 h'(t) = \sum_{j=0}^{N} \frac{g^{(j)}(0)}{j!} \{nj(n \operatorname{Log} t)^{j-1} - (n \operatorname{Log} t)^j\} + O(\{n \operatorname{Log} t\}^{N+1})$$
$$= t^2 \sum_{j=0}^{N} \frac{B_{j+1} n^j}{(j+1)!} \frac{d}{dt}\left(\frac{\operatorname{Log}^j t}{t}\right) + O(\{n \operatorname{Log} t\}^{N+1}). \qquad (14.6)$$

Next, apply the Euler–Maclaurin formula (I3) with 0, $h(t)$, 1, and m playing the roles of n, $f(t)$, α, and β, respectively. Since $h(1) = -\frac{1}{2}$, we find that

$$\sum_{k=2}^{m} h(k) = \int_1^m h(t)\,dt + \frac{1}{4} + \frac{h(m)}{2} + \int_1^m P_1(t) h'(t)\,dt. \qquad (14.7)$$

From (14.6) and (14.7), it follows that

$$\sum_{k=2}^{m} h(k) = \int_1^m h(t)\,dt + \frac{1}{4} + \frac{h(m)}{2}$$
$$+ \sum_{j=0}^{N} \frac{B_{j+1} n^j}{(j+1)!} \int_1^m P_1(t) \frac{d}{dt}\left(\frac{\operatorname{Log}^j t}{t}\right) dt + O(n^{N+1}), \qquad (14.8)$$

where the implied constant is independent of n and m. We now evaluate the integrals on the right side of (14.8). First,

$$\int_1^m h(t)\,dt = \frac{1}{n} \operatorname{Log}\left(\frac{t^n - 1}{t^n \operatorname{Log} t}\right)\Bigg|_1^m$$
$$= \frac{1}{n} \operatorname{Log}\left(\frac{m^n - 1}{m^n}\right) - \frac{\operatorname{Log}\operatorname{Log} m}{n} - \frac{\operatorname{Log} n}{n}. \qquad (14.9)$$

By the Euler–Maclaurin formula (I3),

$$\int_1^m P_1(t)\frac{d}{dt}\left(\frac{\operatorname{Log}^j t}{t}\right)dt$$

$$= \begin{cases} \displaystyle\sum_{k=1}^m \frac{\operatorname{Log}^j k}{k} - \int_1^m \frac{\operatorname{Log}^j t}{t}\,dt - \frac{\operatorname{Log}^j m}{2m} = \sum_{k=1}^m \frac{\operatorname{Log}^j k}{k} - \frac{\operatorname{Log}^{j+1} m}{j+1} \\ \displaystyle\qquad - \frac{\operatorname{Log}^j m}{2m}, \qquad j > 0, \\ \displaystyle\sum_{k=1}^m \frac{1}{k} - \int_1^m \frac{dt}{t} - \frac{1}{2}\left(1 + \frac{1}{m}\right) = \sum_{k=1}^m \frac{1}{k} - \operatorname{Log} m \\ \displaystyle\qquad - \frac{1}{2}\left(1 + \frac{1}{m}\right), \qquad j = 0. \end{cases} \tag{14.10}$$

Using the integral evaluations (14.9) and (14.10) in (14.8) and then letting m tend to ∞, we find that

$$\sum_{k=2}^{\infty} f(k) = \frac{a_0}{n} - \frac{\operatorname{Log} n}{n} + \tfrac{1}{4} + B_1(\gamma - \tfrac{1}{2}) + \sum_{j=1}^{N} \frac{B_{j+1} n^j c_j}{(j+1)!} + O(n^{N+1}),$$

where c_j is defined in (13.1). The asymptotic formula (14.1) now readily follows.

Corollary 1 is a restatement of (14.2).

Corollary 2. *For* $s > 0$,

$$\sum_{k=2}^{\infty} \frac{1}{k^{s+1} \operatorname{Log} k} = -\operatorname{Log} s + C + (1-\gamma)s - \sum_{k=2}^{\infty} \frac{A_{k-1} s^k}{k}, \tag{14.11}$$

where

$$C = \sum_{k=2}^{\infty} \frac{1}{k^2 \operatorname{Log} k} - 1 + \gamma + \sum_{k=2}^{\infty} \frac{A_{k-1}}{k}, \tag{14.12}$$

and where A_k, $1 \le k < \infty$, *is defined in Entry* 13. *Furthermore*, $C = 0.2174630$, $1 - \gamma = 0.4227843$, $-\frac{1}{2}A_1 = -0.0364079$, $-\frac{1}{3}A_2 = 0.001615$, $-\frac{1}{4}A_3 = 0.000086$, *and* $-\frac{1}{5}A_4 = -0.00002$.

Proof. Replacing s by $x + 1$ in (13.2) and integrating over $[1, s]$, we find that, for $s > 0$,

$$-\sum_{k=2}^{\infty} \frac{1}{k^{s+1} \operatorname{Log} k} + \sum_{k=2}^{\infty} \frac{1}{k^2 \operatorname{Log} k} + s - 1$$

$$= \int_1^s \zeta(x+1)\,dx = \operatorname{Log} s + \gamma(s-1) + \sum_{k=2}^{\infty} \frac{A_{k-1}(s^k - 1)}{k}.$$

Hence, (14.11) and (14.12) follow immediately.

The numerical coefficients of s^k, $1 \le k \le 5$, are now found by employing the table of Liang and Todd [1]. The value

$$\sum_{k=2}^{\infty} \frac{1}{k^2 \operatorname{Log} k} = 0.605521788883$$

was calculated by J. R. Hill on his PDP11/34 computer. Using this computation along with the tables of Liang and Todd [1] and the bounds $|A_k| \le 4/(k\pi^k)$, $1 \le k < \infty$, of Berndt [1], we derive the proposed value of C.

Ramanujan's version of Corollary 2 contains some minor discrepancies; his coefficients of s^3 and s^4 are 0.001617 and 0.000085, respectively.

Entry 15. *Let $u > -1$ and $0 < x < 1$. Then*

$$\frac{\varphi_r(x-1) - \varphi_r(-x)}{4\Gamma(r+1)} = -\cos\left(\frac{\pi r}{2}\right) \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{(2\pi k)^{r+1}}.$$

Proof. Recall, from Titchmarsh's treatise [3, p. 37], Hurwitz's formula

$$\zeta(s, a) = 2\Gamma(1-s)\left\{\sin\left(\frac{\pi s}{2}\right) \sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{(2\pi k)^{1-s}} + \cos\left(\frac{\pi s}{2}\right) \sum_{k=1}^{\infty} \frac{\sin(2\pi k a)}{(2\pi k)^{1-s}}\right\}, \tag{15.1}$$

where $\sigma < 1$ and $0 < a \le 1$. By (2.4) and (15.1), the desired formula readily follows.

Entry 16. *Let $u > -1$ and $0 < x < 1$. Then*

$$\frac{\varphi_r(x-1) + \varphi_r(-x) - 2\zeta(-r)}{4\Gamma(r+1)} = \sin\left(\frac{\pi r}{2}\right) \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{(2\pi k)^{r+1}}.$$

Proof. The proof is completely analogous to the previous proof.

Corollary 16(i). *Let p and q be integers with $0 < p < q$. Then if r is any complex number,*

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q} - 1\right) - \varphi_{r-1}\left(-\frac{p}{q}\right)\right\} = -\sin\left(\frac{\pi r}{2}\right) \sum_{j=1}^{q-1} \sin\left(\frac{2\pi j p}{q}\right)\left\{\zeta(r) - \varphi_{-r}\left(\frac{j}{q} - 1\right)\right\}. \tag{16.1}$$

Proof. Using Entry 15 and putting $k = mq + j$, $1 \le j \le q$, $0 \le m < \infty$, we find that, for $r > 1$,

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)-\varphi_{r-1}\left(-\frac{p}{q}\right)\right\}$$

$$= -\sin\left(\frac{\pi r}{2}\right)q^r\sum_{k=1}^{\infty}\frac{\sin(2\pi kp/q)}{k^r}$$

$$= -\sin\left(\frac{\pi r}{2}\right)\sum_{j=1}^{q}\sin\left(\frac{2\pi jp}{q}\right)\sum_{m=0}^{\infty}\frac{1}{(m+j/q)^r}.$$

The result now follows from (2.4) for $r > 1$ and by analytic continuation for all r.

Corollary 16(ii). *Let p and q be integers with $0 < p < q$. For any complex number r,*

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)+\varphi_{r-1}\left(-\frac{p}{q}\right)-2(1-q^{-r})\zeta(1-r)\right\}$$

$$= -\cos\left(\frac{\pi r}{2}\right)\sum_{j=1}^{q-1}\cos\left(\frac{2\pi jp}{q}\right)\left\{\zeta(r)-\varphi_{-r}\left(\frac{j}{q}-1\right)\right\}. \tag{16.2}$$

Proof. The proof is similar to the previous proof, but, in addition, uses the functional equation (4.2).

For the next few results we shall need Ramanujan's extended concept of the Euler numbers from Chapter 6, Section 25. Define

$$E_r^* = \frac{2\Gamma(r)}{(\pi/2)^r}L(r), \tag{17.1}$$

where r is any complex number, and where

$$L(s) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-s}, \qquad \sigma > 0. \tag{17.2}$$

It is well known (e.g., see Davenport's book [1, Chapter 9]) that $L(s)$ can be analytically continued to an entire function.

Entry 17. *For each complex number r,*

$$\varphi_r(-\tfrac{1}{4}) - \varphi_r(-\tfrac{3}{4}) = \frac{2\cos(\pi r/2)E_{r+1}^*}{4^{r+1}}.$$

Proof. Put $x=\frac{1}{4}$ in Entry 15 to get, for $u>-1$,

$$\begin{aligned}\frac{\varphi_r(-\frac{3}{4})-\varphi_r(-\frac{1}{4})}{4\Gamma(r+1)} &= -\cos\left(\frac{\pi r}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{(2\pi k)^{r+1}} \\ &= -\frac{\cos(\pi r/2)}{(2\pi)^{r+1}}L(r+1) \\ &= -\frac{\cos(\pi r/2)E^*_{r+1}}{4^{r+1}2\Gamma(r+1)},\end{aligned}$$

by (17.2) and (17.1). The desired result now follows for $u>-1$. By analytic continuation, the proposed formula is valid for all r.

Corollary. *For* $u<0$,

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)^r=\tfrac{1}{2}\cos\left(\frac{\pi r}{2}\right)E^*_{r+1}.$$

Proof. By Entry 17 and (2.3), with $u<0$,

$$\begin{aligned}\frac{2\cos(\pi r/2)E^*_{r+1}}{4^{r+1}} &= \varphi_r(-\tfrac{1}{4})-\varphi_r(-\tfrac{3}{4}) \\ &= \sum_{k=1}^{\infty}\{(k-\tfrac{3}{4})^r-(k-\tfrac{1}{4})^r\} \\ &= 4^{-r}\sum_{k=0}^{\infty}(-1)^k(2k+1)^r,\end{aligned}$$

and the result follows.

Entry 18. *For each complex number* r,

$$\cos\left(\frac{\pi r}{2}\right)E^*_{1-r}=2L(r)=\left(\frac{\pi}{2}\right)^r\frac{E^*_r}{\Gamma(r)}. \tag{18.1}$$

The equalities in (18.1) yield the functional equation of $L(r)$,

$$L(r)=\cos\left(\frac{\pi r}{2}\right)\left(\frac{\pi}{2}\right)^{r-1}\Gamma(1-r)L(1-r),$$

found, for example, in Davenport's book [1, p. 69].

Proof. We present here Ramanujan's argument.

By (17.1), the "constant" for the series $L(r)$ is

$$L(r)=\frac{(\pi/2)^r}{2\Gamma(r)}E^*_r.$$

But by the last corollary, the "constant" for $L(r)$ is also equal to

$$\tfrac{1}{2}\cos\left(\frac{\pi r}{2}\right)E^*_{1-r}.$$

Since these two constants must be equal, (18.1) follows at once.

Ramanujan's derivation of the next corollary was evidently very similar to his argument for Corollary 4 in Section 4.

Corollary. *We have*

$$\pi\left(\tfrac{1}{2}+\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{2k-1}+\sqrt{2k+1}}\right)=L(\tfrac{3}{2}).$$

Proof. Since the proof is very similar to that of Corollary 4 in Section 4, we shall present only a brief sketch. If we replace $f(x)$ by $f(x+\pi/2)$ in Titchmarsh's text [1, pp. 153–154], we find that

$$\pi^{p+1}+\pi^{p+1}\sum_{k=1}^{\infty}(-1)^k\{(2k+1)^p-(2k-1)^p\}$$
$$=2^{p+2}\cos\left(\frac{\pi p}{2}\right)\Gamma(p+1)L(p+1),$$

where $0<p<1$, after a completely analogous argument. Putting $p=\tfrac{1}{2}$, we complete the proof of the Corollary.

Entry 19(i). *Assume the hypotheses of Corollary* 16(i) *with the additional assumption that q is odd. Then*

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)-\varphi_{r-1}\left(-\frac{p}{q}\right)\right\}$$
$$=\sin\left(\frac{\pi r}{2}\right)\sum_{j=1}^{(q-1)/2}\sin\left(\frac{2\pi jp}{q}\right)\left\{\varphi_{-r}\left(\frac{j}{q}-1\right)-\varphi_{-r}\left(-\frac{j}{q}\right)\right\}.$$

Proof. On the right side of (16.1) replace j by $q-j$ in that part of the sum with $(q+1)/2\le j\le q-1$.

Entry 19(ii). *Suppose that all hypotheses of Corollary* 16(ii) *hold. Assume also that q is odd. Then*

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)+\varphi_{r-1}\left(-\frac{p}{q}\right)-2\zeta(1-r)\right\}$$
$$=\cos\left(\frac{\pi r}{2}\right)\sum_{j=1}^{(q-1)/2}\cos\left(\frac{2\pi jp}{q}\right)\left\{\varphi_{-r}\left(\frac{j}{p}-1\right)+\varphi_{-r}\left(-\frac{j}{q}\right)\right\}.$$

Proof. Using (16.2), proceed in the same fashion as in the previous proof. In addition, the functional equation (4.2) must be employed.

Ramanujan's version of Entry 19(ii) is incorrect (pp. 86, 87).

Corollary 1. *Let $u > 0$ and suppose that $0 \le x < 1$. Then*

$$\frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\varphi_r(-x) = \sum_{k=1}^{\infty} \frac{\sin(\pi kx)\cos(\pi kx + \pi r/2)}{k^{r+1}}.$$

Proof. By (2.4), (4.2), and (15.1), we find that for $u > 0$ and $0 \le x < 1$,

$$\begin{aligned}\frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\varphi_r(-x) &= \frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\{\zeta(-r) - \zeta(-r, 1-x)\}\\ &= -\frac{1}{2}\sin\left(\frac{\pi r}{2}\right)\zeta(r+1) + \frac{1}{2}\sin\left(\frac{\pi r}{2}\right)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{r+1}}\\ &\quad + \frac{1}{2}\cos\left(\frac{\pi r}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^{r+1}}\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^{r+1}}\left\{-\sin\left(\frac{\pi r}{2}\right) + \sin\left(2\pi kx + \frac{\pi r}{2}\right)\right\}\\ &= \sum_{k=1}^{\infty}\frac{\sin(\pi kx)\cos(\pi kx + \pi r/2)}{k^{r+1}},\end{aligned}$$

upon using the identity $-\sin A + \sin(A + 2B) = 2\sin B\cos(A + B)$. This completes the proof.

Corollary 2. *If* $0 < x < 1$, *then*

$$\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{k+x}} - \frac{1}{\sqrt{k+1-x}}\right) = 2\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{\sqrt{k}}.$$

Proof. Set $r = -\frac{1}{2}$ in Entry 15 and use (2.3).

Entry 20. *If r is any complex number, then*

$$\frac{(6\pi)^r}{2\sqrt{3}\Gamma(r)}\{\varphi_{r-1}(-\tfrac{1}{3}) - \varphi_{r-1}(-\tfrac{2}{3})\} = \sin\left(\frac{\pi r}{2}\right)\{\varphi_{-r}(-\tfrac{1}{3}) - \varphi_{-r}(-\tfrac{2}{3})\}.$$

Proof. Put $p = 1$ and $q = 3$ in Entry 19(i), and the result follows.

Section 21 appears to have no relation to the other material in Chapter 7. In Entry 21, Ramanujan writes

$$\varphi_{\infty}\left(\frac{nx}{1+x}\right) = (1+x)^{-n}\sum_{k=0}^{\infty}\binom{n}{k}\varphi(k)x^k, \tag{21.1}$$

where

$$\varphi_r(x) = \varphi_{r-1}(x) - 1 + \exp\left(\frac{nP_{r-1}\varphi_{r-1}^r(x)}{r!}\right),$$

with $\varphi_1(x) = \varphi(x)$, and where

$$P_r = \sum_{k=1}^{\infty} (-1)^{k+1} k^r x^k.$$

We have been unable to discern the meaning of this result, since the recursively defined functions $\varphi_r(x)$ have not been connected with (21.1) in any way. We shall regard (21.1) as the definition of φ_∞. Setting $u = nx/(1+x)$ and $p = u/n$, we find that (21.1) becomes

$$\varphi_\infty(u) = \sum_{k=0}^{\infty} \binom{n}{k} \varphi(k) p^k (1-p)^{n-k}. \tag{21.2}$$

In the following corollary, Ramanujan gives a formula for $\varphi_\infty(u)$ in terms of the derivatives $\varphi^{(j)}(u)$, $0 \le j < \infty$. Note that $\varphi_\infty(u)$ is the expected value of $\varphi(u)$ if u denotes a random variable with binomial distribution $b(n, k; p)$. Ramanujan alludes to Entry 10 of Chapter 3, where he gives a formula for the expected value of $\varphi(u)$ in terms of $\varphi^{(j)}(u)$, $0 \le j < \infty$, where u denotes a Poisson random variable. However, the latter result appears to be considerably deeper than the present corollary.

Corollary. *Let u and n be fixed, where $0 \le u \le n$ and n is an integer. Let $\varphi(z)$ be analytic in a disc centered at u and containing the segment $[0, n]$. If $\varphi_\infty(u)$ is defined by (21.2), then*

$$\varphi_\infty(u) = \sum_{j=0}^{\infty} \frac{\varphi^{(j)}(u)}{j!} \sum_{k=0}^{\infty} \binom{n}{k} (k-u)^j p^k (1-p)^{n-k}. \tag{21.3}$$

Proof. Expanding $\varphi(z)$ in its Taylor series about u, we find that

$$\varphi_\infty(u) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{\infty} \frac{\varphi^{(j)}(u)}{j!} (k-u)^j p^k (1-p)^{n-k}.$$

The equality (21.3) now follows by inverting the order of summation.

Observe that the Corollary even holds when n is an arbitrary positive number, provided that $p < \frac{1}{2}$ and φ is a polynomial. It would be interesting to find more general conditions under which (21.3) holds.

Entry 22. *Let $A_1 = 0$. For each nonnegative integer r, $r \ne 1$, set $A_r = \{1 + (-1)^r\}\zeta(r)$. If n is a natural number, then*

$$\sum_{k=1}^{\infty} \frac{1}{k^n (k+1)^n} = \sum_{k=0}^{n} A_{n-k} \binom{-n}{k}.$$

Proofs of Entry 22 have been given by Glaisher [3], Kesava Menon [2], and Djoković [1]. Entry 22 is identical with Entry 35 in Chapter 9. The following example is the special case $n = 3$ of Entry 22.

Example. $$\sum_{k=1}^{\infty} \frac{1}{k^3(k+1)^3} = 10 - \pi^2.$$

Entry 23 offers Stirling's well-known asymptotic expansion of $\operatorname{Log}\Gamma(z+1)$. (E.g., see Whittaker and Watson's treatise [1, p. 252].)

Entry 23. *Let* $|\arg z| < \pi$. *Then as* $|z|$ *tends to* ∞,

$$\operatorname{Log}\Gamma(z+1) \sim (z+\tfrac{1}{2})\operatorname{Log} z - z + \tfrac{1}{2}\operatorname{Log}(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}}.$$

Ramanujan remarks that $\frac{1}{2}\operatorname{Log}(2\pi) = 0.918938533204673$, which is correct (Abramowitz and Stegun [1, p. 3]).

We quote the following corollary exactly as it appears in the notebooks, p. 88. This approximation for the gamma function is remindful of Corollaries 6–8 in Section 4.

Corollary. *When* x *is great* $e^x\Gamma(x+1)/x^x = \sqrt{2\pi x + \pi/3}$ *nearly.*

Proof. From the familiar asymptotic expansion (I6) for $\Gamma(x)$, as x tends to ∞,

$$\frac{e^x\Gamma(x+1)}{x^x} \sim \sqrt{2\pi x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right). \tag{23.1}$$

But, on the other hand,

$$\sqrt{2\pi x + \frac{\pi}{3}} = \sqrt{2\pi x}\sqrt{1 + \frac{1}{6x}} = \sqrt{2\pi x}\left(1 + \frac{1}{12x} - \frac{1}{288x^2} + \cdots\right). \tag{23.2}$$

Thus, Ramanujan's approximation is reasonable, but observe that the coefficients of x^{-2} in (23.1) and (23.2) are of opposite sign.

Entry 24 and its corollary are restatements of Corollaries 3 and 2, respectively, of Section 4.

Entry 25. *For every complex number* z *and positive integer* n,

$$\prod_{k=1}^{n} \Gamma\left(\frac{z+k}{n}\right) = (2\pi)^{(n-1)/2} n^{-z-1/2}\Gamma(z+1).$$

Entry 25 is a version of Gauss's famous multiplication theorem for the gamma function (Whittaker and Watson [1, p. 240]). Corollary 1 is the special case $z=0$ of Entry 25.

Corollary 2. $\Gamma(\frac{2}{3})=\sqrt{\Gamma(\frac{5}{6})\sqrt[3]{2}\sqrt[4]{\pi/3}}$.

Proof. Put $n=2$ and $z=-\frac{1}{3}$ in Entry 25 to get $\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})=\sqrt{\pi}2^{1/3}\Gamma(\frac{2}{3})$. By Corollary 3 of Section 4, $\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})=2\pi/\sqrt{3}$. Combining these two equalities, we achieve the desired result.

Corollary 3. *For every complex number z,*

$$\Gamma(z+1)=\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{z}{2}+1\right)2^{z}\pi^{-1/2}.$$

Corollary 3 is Legendre's duplication formula and is the special case $n=2$ of Gauss's multiplication formula, Entry 25.

Corollary 4. *Let* $|\arg z|<\pi$. *Then as* $|z|$ *tends to* ∞,

$$\operatorname{Log}\Gamma(z+\tfrac{1}{2})\sim z\operatorname{Log}z-z+\tfrac{1}{2}\operatorname{Log}(2\pi)+\sum_{k=1}^{\infty}\frac{B_{2k}(2^{1-2k}-1)}{2k(2k-1)z^{2k-1}}.$$

Proof. Replacing z by $2z$ in Corollary 3, we find that

$$\Gamma(z+\tfrac{1}{2})=\frac{\sqrt{\pi}\,\Gamma(2z+1)}{2^{2z}\Gamma(z+1)}.$$

Take logarithms on both sides and apply Entry 23.

Ramanujan inadvertently multiplied the infinite series above by -1.

The Maclaurin series in Entry 26 is well known (Abramowitz and Stegun [1, p. 256]).

Entry 26. *For* $|z|<1$,

$$\operatorname{Log}\Gamma(z+1)=-\gamma z+\sum_{k=2}^{\infty}\frac{\zeta(k)(-z)^{k}}{k},$$

where γ *denotes Euler's constant.*

Corollary. *For* $|z|<1$,

$$\begin{aligned}\operatorname{Log}\{\tfrac{1}{2}\Gamma(z+3)\}&=0.9227843351z+0.1974670334z^{2}\\&\quad-0.0256856344z^{3}+0.0049558084z^{4}\\&\quad-0.0011355510z^{5}+0.0002863437z^{6}\\&\quad-0.0000766825z^{7}+0.0000213883z^{8}\\&\quad-0.0000061409z^{9}+0.0000018013z^{10}+\dots. \qquad (26.1)\end{aligned}$$

Proof. Using Entry 26, we find that, for $|z| < 1$,

$$\begin{aligned}\operatorname{Log}\{\tfrac{1}{2}\Gamma(z+3)\} &= \operatorname{Log}(\tfrac{1}{2}z+1) + \operatorname{Log}(z+1) + \operatorname{Log}\Gamma(z+1) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{z}{2}\right)^k + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}z^k}{k} \\ &\quad - \gamma z + \sum_{k=2}^{\infty} \frac{\zeta(k)(-z)^k}{k}.\end{aligned}$$

The given numerical values for the coefficients of z^k, $1 \le k \le 10$, now follow by direct calculation. Numerical values of $\zeta(k)$, $2 \le k \le 10$, may be found in the tables of Abramowitz and Stegun [1, p. 811] or Dwight [1, p. 224].

In Ramanujan's formulation of (26.1), he replaces the tenth and all succeeding terms by the single expression $0.0000054047z^{10}/(3+z)$.

Our calculations below were determined from the values of $\Gamma(\frac{2}{3})$, $\Gamma(\frac{5}{6})$, and $\Gamma(\frac{9}{10})$ found in Fransén and Wrigge's tables [1]. Ramanujan inexplicably gives the value 0.5341990853 for $\operatorname{Log}\Gamma(\frac{2}{3})$.

Example 1. $\operatorname{Log}\Gamma(\frac{2}{3}) = 0.3031502752.$

Example 2. $\operatorname{Log}\Gamma(\frac{5}{6}) = 0.1211436313.$

Example 3. $\operatorname{Log}\Gamma(\frac{9}{10}) = 0.0663762397.$

Entry 27(i). *Suppose that n is a natural number and that $|z| > n$. Then*

$$2\pi z \prod_{k=1}^{\infty}\left(1 + \frac{z^2}{(n+k)^2}\right) = 2\left(\frac{n!}{z^n}\right)^2 \sinh(\pi z) \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k \varphi_{2k}(n)}{kz^{2k}}\right). \tag{27.1}$$

Proof. Using Euler's definition of the gamma function (Whittaker and Watson [1, p. 237])

$$\Gamma(z) = \lim_{k\to\infty} \frac{(k-1)!\,k^z}{z(z+1)\cdots(z+k-1)},$$

which is also Example 2, Section 4 of Chapter 6, we find that

$$\begin{aligned}&\frac{\Gamma^2(n+1)}{\Gamma(n+1+iz)\Gamma(n+1-iz)} \\ &\qquad = \lim_{k\to\infty} \frac{\{(n+1)^2+z^2\}\{(n+2)^2+z^2\}\cdots\{(n+k)^2+z^2\}}{(n+1)^2(n+2)^2\cdots(n+k)^2} \\ &\qquad = \prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right).\end{aligned} \tag{27.2}$$

On the other hand,

$$\frac{2\pi z\Gamma^2(n+1)}{\Gamma(n+1+iz)\Gamma(n+1-iz)}$$

$$= \frac{2\pi z\Gamma^2(n+1)}{z^{2n}(1+1^2/z^2)(1+2^2/z^2)\cdots(1+n^2/z^2)\Gamma(1+iz)\Gamma(1-iz)}. \tag{27.3}$$

Using the Maclaurin series for $\text{Log}(1+y)$ with $y = k^2/z^2$, $1 \le k \le n$, we find that

$$\prod_{k=1}^{n}\left(1+\frac{k^2}{z^2}\right)^{-1} = \exp\left(-\sum_{k=1}^{n}\text{Log}\left(1+\frac{k^2}{z^2}\right)\right)$$

$$= \exp\left(\sum_{j=1}^{\infty}\frac{(-1)^j\varphi_{2j}(n)}{jz^{2j}}\right), \tag{27.4}$$

provided that $|z| > n$. Also, by Corollary 3 of Section 4,

$$\frac{2\pi z}{\Gamma(1+iz)\Gamma(1-iz)} = \frac{2\pi}{i\Gamma(iz)\Gamma(1-iz)} = 2\sinh(\pi z). \tag{27.5}$$

Now substitute (27.4) and (27.5) into (27.3). Comparing the resulting equality with (27.2), we readily deduce (27.1).

In Ramanujan's formulations of Entries 27(i) and (ii), pp. 89, 90, instead of $2\sinh(\pi x)$, there appears $e^{\pi x} - e^{-\pi x}\theta$, but $e^{-\pi x}\theta$ is struck out. In a footnote, which is also struck out, Ramanujan says that "$\theta = \cos 2\pi n$ exactly or very nearly according as $2n$ is an integer or not." A two-line solution to Entry 27(i) is also crossed out.

Entry 27(ii). *Under the same hypotheses as Entry* 27(i), *we have*

$$2\pi(z^2+n^2)^{n+1/2}\prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right)$$

$$= 2(n!)^2\sinh(\pi z)\exp\left(2n - 2z\tan^{-1}\left(\frac{n}{z}\right) + \sum_{j=1}^{\infty}\frac{(-1)^j B_{2j}S_{2j}}{jz^{2j-1}}\right), \tag{27.6}$$

where

$$S_{2j} = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+2j-1)!}{(2j-1)!\,(2k+1)!}\left(\frac{n}{z}\right)^{2k+1}, \qquad j \ge 1.$$

Proof. First,

$$(z^2+n^2)^{n+1/2}/z = z^{2n}e^{(n+1/2)\,\text{Log}(1+n^2/z^2)}. \tag{27.7}$$

Multiplying both sides of (27.1) by $(z^2+n^2)^{n+1/2}/z$ and utilizing (27.7), we find that, for $|z|>n$,

$$2\pi(z^2+n^2)^{n+1/2}\prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right)$$

$$=2(n!)^2\sinh(\pi z)\exp\left((n+\tfrac{1}{2})\sum_{k=1}^{\infty}\frac{(-1)^{k+1}n^{2k}}{kz^{2k}}+\sum_{k=1}^{\infty}\frac{(-1)^k\varphi_{2k}(n)}{kz^{2k}}\right). \tag{27.8}$$

Comparing (27.6) with (27.8), we see that it remains to show that

$$(n+\tfrac{1}{2})\sum_{k=1}^{\infty}\frac{(-1)^{k+1}n^{2k}}{kz^{2k}}+\sum_{k=1}^{\infty}\frac{(-1)^k\varphi_{2k}(n)}{kz^{2k}}$$

$$=2n-2z\tan^{-1}\left(\frac{n}{z}\right)+\sum_{j=1}^{\infty}\frac{(-1)^jB_{2j}S_{2j}}{jz^{2j-1}}. \tag{27.9}$$

By Entry 1 and the remarks prior to (1.6), since $\zeta(-2k)=0$, $k\geq 1$,

$$\sum_{k=1}^{\infty}\frac{(-1)^k\varphi_{2k}(n)}{kz^{2k}}=\sum_{k=1}^{\infty}\frac{(-1)^kn^{2k+1}}{k(2k+1)z^{2k}}+\sum_{k=1}^{\infty}\frac{(-1)^kn^{2k}}{2kz^{2k}}$$

$$+\sum_{k=1}^{\infty}\frac{(-1)^k}{kz^{2k}}\sum_{j=1}^{k}\frac{B_{2j}(2k)!\,n^{2k-2j+1}}{(2j)!\,(2k-2j+1)!}. \tag{27.10}$$

Now a short calculation shows that

$$\sum_{k=1}^{\infty}\frac{(-1)^kn^{2k+1}}{k(2k+1)z^{2k}}+\sum_{k=1}^{\infty}\frac{(-1)^kn^{2k}}{2kz^{2k}}+(n+\tfrac{1}{2})\sum_{k=1}^{\infty}\frac{(-1)^{k+1}n^{2k}}{kz^{2k}}$$

$$=2z\sum_{k=1}^{\infty}\frac{(-1)^{k+1}n^{2k+1}}{(2k+1)z^{2k+1}}=2n-2z\tan^{-1}\left(\frac{n}{z}\right).$$

Thus, we only need yet to examine the double sum in (27.10). Inverting the order of summation by absolute convergence, we find that this double series becomes

$$\sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}\sum_{k=j}^{\infty}\frac{(-1)^k(2k)!\,n^{2k-2j+1}}{k(2k-2j+1)!\,z^{2k}}$$

$$=\sum_{j=1}^{\infty}\frac{(-1)^jB_{2j}}{z^{2j-1}}\sum_{\mu=0}^{\infty}\frac{(-1)^\mu(2\mu+2j)!}{(2j)!\,(\mu+j)(2\mu+1)!}\left(\frac{n}{z}\right)^{2\mu+1}.$$

After a slight amount of simplification, the double series above is easily seen to be equal to the series on the right side of (27.9). This completes the proof of (27.9) and hence of (27.6).

Entry 27(iii). *Let n be a positive integer and suppose that $x > 0$. Write $r^2 = n^2 + x^2$ with $r > 0$ and put $\beta = \tan^{-1}(x/n)$. Then as x tends to ∞,*

$$\operatorname{Log}\left\{2\pi(x^2+n^2)^{n-1/2}\prod_{k=0}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right)\right\}$$
$$\sim 2\operatorname{Log}\Gamma(n) + 2n + 2x\beta - \sum_{k=1}^{\infty}\frac{B_{2k}\cos\{(2k-1)\beta\}}{k(2k-1)r^{2k-1}}.$$

Proof. Using (27.2) and Entry 23, we find that, as x tends to ∞,

$$\begin{aligned}
&\operatorname{Log}\left\{2\pi(x^2+n^2)^{n-1/2}\prod_{k=0}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right)\right\}\\
&\qquad= \operatorname{Log}\left\{\frac{2\pi\{(n-1)!\}^2(x^2+n^2)^{n+1/2}}{\Gamma(n+1+ix)\Gamma(n+1-ix)}\right\}\\
&\qquad= \operatorname{Log}(2\pi) + 2\operatorname{Log}\Gamma(n) + (2n+1)\operatorname{Log} r\\
&\qquad\quad - \operatorname{Log}\Gamma(n+1+ix) - \operatorname{Log}\Gamma(n+1-ix)\\
&\qquad\sim \operatorname{Log}(2\pi) + 2\operatorname{Log}\Gamma(n) + (2n+1)\operatorname{Log} r\\
&\qquad\quad - (n+ix+\tfrac{1}{2})\operatorname{Log}(n+ix)\\
&\qquad\quad - (n-ix+\tfrac{1}{2})\operatorname{Log}(n-ix) - \operatorname{Log}(2\pi) + 2n\\
&\qquad\quad - \sum_{k=1}^{\infty}\frac{B_{2k}\{(n+ix)^{2k-1}+(n-ix)^{2k-1}\}}{2k(2k-1)r^{2(2k-1)}}.
\end{aligned}$$

The desired result now follows since $n + ix = re^{i\beta}$.

We are very grateful to J. W. Wrench, Jr. for supplying several references.

CHAPTER 8

Analogues of the Gamma Function

The first 14 sections of Chapter 8 comprise but $4\frac{1}{2}$ of the 12 pages in this chapter. Initial results are concerned with partial sums of the harmonic series and the logarithmic derivative $\psi(x)$ of the gamma function. As might be expected, most of these results are very familiar. Ramanujan actually does not express his formulas in terms of $\psi(x)$ but instead in terms of $\varphi(x) = \sum_{k=1}^{x} 1/k$. As in Chapter 6, Ramanujan really intends $\varphi(x)$ to be interpreted as $\psi(x+1)+\gamma$, *for all real* x, where γ denotes Euler's constant. These 14 sections also contain several evaluations of elementary integrals of rational functions. Certain of these integrals are connected with an interesting series $\sum_{k=1}^{\infty} 1/\{(kx)^3 - kx\}$, which Ramanujan also examined in Chapter 2.

Sections 17–24 constitute the focus of Chapter 8 and are rather more mathematically sophisticated than the initial 14 sections. Essentially all of the results in these last sections appear to be new. Ramanujan studies several intriguing analogues of the gamma function. In particular, he derives numerous analogues of Stirling's formula, Gauss's multiplication theorem, and Kummer's formula for $\operatorname{Log}\Gamma(x)$. Ramanujan continues this study in Sections 27–30 of Chapter 9. The results in especially Chapter 9 are related to a generalization of $\Gamma(x)$ studied by Bendersky [1] and Büsing [1]. Post [1] has considered another type of generalization of $\Gamma(x)$, but this appears to be unrelated to Ramanujan's work. One of the difficulties in examining Sections 17–24 is that Ramanujan initially defines an analogue $\varphi(x)$ of $\operatorname{Log}\Gamma(x+1)$ for only positive integral values of x by a finite sum with upper index x. He then develops properties of an analytic extension of $\varphi(x)$, but he invariably does not share with us his more general definition of $\varphi(x)$.

Recalling Lerch's formula for $\operatorname{Log}\Gamma(x)$ in terms of the Hurwitz zeta-function, for some of Ramanujan's extensions, we have defined $\varphi(x)$ by similar types of formulas involving the analytic continuation of the Hurwitz zeta-function. Although Ramanujan had no firm grasp of analytic continuation, we think that Ramanujan somewhat nonrigorously used an approach like this. It is interesting to note that this circle of ideas has independently been observed by H. M. Stark [2]. However, whereas Ramanujan was primarily interested in analogues of Log $\Gamma(x)$, Stark's interest is in using the Hurwitz zeta-function and its analytic continuation to determine values of zeta and L-functions at nonpositive integers. It also might be remarked that Berndt [4] and Milnor [1] have shown that the main properties of the gamma function may be established by using only knowledge about the Hurwitz zeta-function.

If $L(s, \chi)$ denotes the classical Dirichlet L-function associated with the character χ, then there is a classical formula for $L'(0, \chi)$ that depends upon Lerch's formula for Log $\Gamma(x)$. Analogues have recently been established for other L-functions, e.g., p-adic L-functions, and analogous of Lerch's formula naturally arise. (See papers of Ferrero and Greenberg [1], Gross and Koblitz [1], Moreno [1], and Shintani [1].) It is hoped that some of the analogues in Chapters 8 and 9 might have similar applications.

Throughout this chapter, the real part of a complex variable s shall be denoted by σ.

Ramanujan begins Chapter 8 by stating "$B_r^* \cos(\frac{1}{2}\pi r)/r + 1/r$, when r vanishes, is a finite quantity which is invariably denoted by c_0; it is the constant of S_1 ...." In order to understand this claim, we must recall Ramanujan's definition of Bernoulli numbers. Interpolating Euler's formula for $\zeta(2n)$, Ramanujan, in (25.1) of Chapter 5, defined extended Bernoulli numbers B_r^* by

$$B_r^* = \frac{2\Gamma(r+1)}{(2\pi)^r}\,\zeta(r), \tag{1.1}$$

where now we assume that r is any complex number. In Chapter 7, Entry 4, Ramanujan showed that $\zeta(r) = B_{1-r}^* \sin(\frac{1}{2}\pi r)/(1-r)$, for any complex number r. Thus, Ramanujan's opening remark may be expressed as

$$\lim_{r\to 0}\left\{\zeta(1-r) + \frac{1}{r}\right\} = c_0.$$

Of course, this is well known, and $c_0 = \gamma$, Euler's constant. In Ramanujan's notation, $S_1 = \sum 1/k$. We remark that in Chapter 7 Ramanujan derived the full Laurent series of $\zeta(1-r)$ about $r = 0$ in Entry 13.

Next, Ramanujan remarks that $\gamma = 0.577215664901533$ and $e^{-\gamma} = 0.56145948356$. The former calculation is correct, while in the latter, the last recorded digit should be 7. (See Abramowitz and Stegun [1, pp. 3, 2], respectively.)

Entry 2. *As x tends to ∞,*

$$\sum_{k=1}^{x} \frac{1}{k} \sim \text{Log}\, x + \gamma - \sum_{k=1}^{\infty} \frac{B_k}{kx^k}.$$

This asymptotic expansion is well known and is due to Euler. For the usual proof, via the Euler–Maclaurin summation formula, see Bromwich's book [1, pp. 324, 325].

Entry 3. *If x is real, then*

$$\sum_{k\le x}\frac{1}{k}=\sum_{k=1}^{\infty}\left\{\frac{1}{k}-\frac{1}{k+x}\right\}=x\sum_{k=1}^{\infty}\frac{1}{k(k+x)}. \tag{3.1}$$

Now if x is a positive integer, Entry 3 is a complete triviality. In essence, Ramanujan *defines* a function $\varphi(x)$ ($\sum_{k\le x} 1/k$ in Ramanujan's notation) for all real x by (3.1). Ramanujan adopts this convention in Chapter 7, equation (2.3), as well.

Entry 4. *For $|x|<1$,*

$$\sum_{k\le x}\frac{1}{k}=\sum_{k=1}^{\infty}(-1)^{k+1}\zeta(k+1)x^k. \tag{4.1}$$

This result again needs interpretation. From the Weierstrass product of the gamma function (see, e.g., Whittaker and Watson's treatise [1, p. 247]),

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)}+\gamma=\sum_{k=1}^{\infty}\left\{\frac{1}{k}-\frac{1}{k+x}\right\}, \tag{4.2}$$

where x is any complex number. In particular, if x is a positive integer, then

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)}+\gamma=\sum_{k=1}^{x}\frac{1}{k}.$$

Thus, in place of the left side of (4.1), it is more natural to take the left side of (4.2), which is in agreement with (3.1), and with this alteration (4.1) is valid. In fact, Entry 4 is identical to Example 2 of Section 6 in Chapter 6. See also (5.4) and (5.11) in Chapter 6.

Entry 5. *For any complex number x,*

$$\sum_{k=1}^{\infty}\left\{\frac{1}{k-1+x}-\frac{1}{k-x}\right\}=\pi\cot(\pi x). \tag{5.1}$$

Entry 5 simply records the familiar partial fraction decomposition of $\cot(\pi x)$. (Ramanujan incorrectly multiplies the right side of (5.1) by -1). A very short proof may be given as follows. Logarithmically differentiate the reflection formula $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$ to obtain

$$\frac{\Gamma'(x)}{\Gamma(x)}-\frac{\Gamma'(1-x)}{\Gamma(1-x)}=-\pi\cot(\pi x). \tag{5.2}$$

Then employ (4.2).

In the sequel, we shall employ the familiar notation $\psi(x)=\Gamma'(x)/\Gamma(x)$, whereas Ramanujan expresses his results in terms of

$$\varphi(x)=\frac{\Gamma'(x+1)}{\Gamma(x+1)}+\gamma$$

(which he regards as $\sum_{k\le x} 1/k$).

Entry 6. *Let n denote a positive integer and suppose that x is any complex number. Then*

$$n\psi(x+1)-\sum_{k=0}^{n-1}\psi\left(\frac{x+k+1}{n}\right)=n \operatorname{Log} n.$$

This result is well known (see Gradshteyn and Ryzhik [1, p. 945]) and arises from the logarithmic differentiation of Gauss's multiplication theorem.

Corollary 1. *Let* $|\arg x|<\pi$. *Then as* $|x|$ *tends to* ∞,

$$\psi(x+\tfrac{1}{2})\sim \operatorname{Log} x+\sum_{k=1}^{\infty}\frac{B_{2k}(1-2^{1-2k})}{2kx^{2k}}.$$

Proof. Differentiate both sides in Corollary 4, Section 25 of Chapter 7.

Corollary 2. *If n is a positive integer, then*

$$\sum_{k=1}^{n-1}\psi\left(\frac{k}{n}\right)=-(n-1)\gamma-n \operatorname{Log} n. \tag{6.1}$$

Proof. Setting $z=x+1$ in Entry 6 and letting z tend to 0, we find that

$$\sum_{k=1}^{n-1}\psi\left(\frac{k}{n}\right)=\lim_{z\to 0}\left\{n\psi(z)-\psi\left(\frac{z}{n}\right)\right\}-n \operatorname{Log} n.$$

By using (4.2), we easily find the foregoing limit to be $-(n-1)\gamma$, and this completes the proof.

Corollary 3. *We have*

(i) $$\psi(\tfrac{1}{2})=-\gamma-2 \operatorname{Log} 2,$$

(ii) $$\psi(\tfrac{1}{3})=-\gamma-\tfrac{3}{2} \operatorname{Log} 3-\frac{\pi}{2\sqrt{3}},$$

(iii) $$\psi(\tfrac{1}{4})=-\gamma-3 \operatorname{Log} 2-\frac{\pi}{2},$$

(iv) $$\psi(\tfrac{1}{6})=-\gamma-2 \operatorname{Log} 2-\tfrac{3}{2} \operatorname{Log} 3-\frac{\pi\sqrt{3}}{2},$$

and

(v) $$3\psi(\tfrac{1}{2})-2\psi(\tfrac{1}{4})=-\gamma+\pi.$$

Proof. Part (i) follows from putting $n = 2$ in Corollary 2. Part (ii) follows from Corollary 2 with $n = 3$ and (5.2) with $x = 1/3$. The proofs of (iii) and (iv) are similar.

Corollary 4. *If n is a positive integer, then*

$$\sum_{k=1}^{n} \psi\left(\frac{2k-1}{2n}\right) = -n\gamma - n \operatorname{Log}(4n).$$

Proof. Replacing n by $2n$ in (6.1), we find that

$$\sum_{k=1}^{2n-1} \psi\left(\frac{k}{2n}\right) = -(2n-1)\gamma - 2n \operatorname{Log}(2n). \tag{6.2}$$

Subtracting (6.1) from (6.2), we achieve Corollary 4.

Entry 7. *If x is a positive integer and a and b are arbitrary complex numbers, then*

$$\psi\left(\frac{a}{b} + x + 1\right) - \psi\left(\frac{a}{b} + 1\right) = b \sum_{k=1}^{x} \frac{1}{a + bk}.$$

Proof. This result is an easy consequence of (4.2).

Entry 8. *If a and b are arbitrary complex numbers, then*

$$\psi\left(\frac{a+2b}{2b}\right) - \psi\left(\frac{a+b}{2b}\right) = 2b \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{a+bk}.$$

Proof. The desired formula follows easily from (4.2).

Entry 9. *If* Re $x > 0$, *then*

$$\psi\left(\frac{1}{2x} + 1\right) = \psi\left(\frac{1}{x} + 1\right) - \operatorname{Log} 2 + x \int_0^1 \frac{u^x}{1+u^x}\, du. \tag{9.1}$$

Proof. By (4.2),

$$\psi\left(\frac{1}{2x} + 1\right) + \gamma = \lim_{N\to\infty} \sum_{k=1}^{N} \left\{\frac{1}{k} - \frac{2x}{2kx+1}\right\}$$

and

$$\psi\left(\frac{1}{x} + 1\right) + \gamma = \lim_{N\to\infty} \sum_{k=1}^{2N} \left\{\frac{1}{k} - \frac{x}{kx+1}\right\}.$$

Hence,

$$\begin{aligned}\psi\left(\frac{1}{2x} + 1\right) - \psi\left(\frac{1}{x} + 1\right) &= \lim_{N\to\infty} \left\{-\sum_{k=N+1}^{2N} \frac{1}{k} + \sum_{k=1}^{2N} \frac{(-1)^{k+1}x}{kx+1}\right\} \\ &= -\operatorname{Log} 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x}{kx+1}.\end{aligned} \tag{9.2}$$

Now, if Re $x > 0$,

$$\int_0^1 \frac{u^x}{1+u^x}\,du = \int_0^1 \sum_{k=1}^{\infty} (-1)^{k+1} u^{kx}\,du = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{kx+1}. \tag{9.3}$$

Combining (9.2) and (9.3), we obtain (9.1).

Entry 10. *If* Re $x < 0$, *then*

$$\psi\left(1-\frac{1}{x}\right) + \gamma = -x \int_0^1 \frac{1-u}{u(u^x-1)}\,du.$$

Proof. If $0 \le u < 1$ and Re $x < 0$, then $|u^{-x}| < 1$, and so

$$\begin{aligned} -x \int_0^1 \frac{1-u}{u(u^x-1)}\,du &= x \int_0^1 \sum_{k=1}^{\infty} \{u^{-kx} - u^{-kx-1}\}\,du \\ &= \sum_{k=1}^{\infty} \left\{\frac{x}{1-kx} + \frac{1}{k}\right\}. \end{aligned}$$

Appealing to (4.2), we complete the proof.

Observe that the integral in Entry 10 diverges if Re $x > 0$. In the integrand Ramanujan has incorrectly squared the expression $1-u$ in the notebooks, p. 92.

For brevity, in the sequel we shall put

$$\varphi(x) = 1 + \sum_{k=1}^{\infty} \frac{2}{(kx)^3 - kx}. \tag{11.1}$$

Entry 11. *If x is any complex number, then*

$$\psi\left(\frac{1}{x}\right) + \psi\left(1-\frac{1}{x}\right) = -2\gamma - x\varphi(x).$$

Proof. Using (4.2) below, we get

$$\begin{aligned} \psi\left(\frac{1}{x}\right) + \psi\left(1-\frac{1}{x}\right) &= -x + \psi\left(1+\frac{1}{x}\right) + \psi\left(1-\frac{1}{x}\right) \\ &= -2\gamma - x + \sum_{k=1}^{\infty} \left\{\frac{2}{k} - \frac{x}{kx+1} - \frac{x}{kx-1}\right\}, \end{aligned}$$

from which the desired result follows after a slight amount of simplification.

Entry 12. *If* Re $x > 1$, *then*

$$\int_0^1 \frac{u^{x-2}(1-u)^2}{1-u^x}\,du = \varphi(x) - 1.$$

Proof. We have

$$\int_0^1 \frac{u^{x-2}(1-u)^2}{1-u^x}\,du = \int_0^1 \sum_{k=1}^{\infty} \{u^{kx-2} - 2u^{kx-1} + u^{kx}\}\,du$$
$$= \sum_{k=1}^{\infty} \left\{\frac{1}{kx-1} - \frac{2}{kx} + \frac{1}{kx+1}\right\},$$

and the result follows.

Entry 13. *If* Re $x > 0$, *then*

$$\varphi(2x) - \tfrac{1}{2}\varphi(x) - \frac{1}{x}\operatorname{Log} 2 = 1 - \frac{\pi}{2x}\csc\left(\frac{\pi}{x}\right) - \int_0^1 \frac{u^x}{1+u^x}\,du. \tag{13.1}$$

Proof. By Entry 11 and (5.2),

$$\varphi(x) = -\frac{1}{x}\left\{\psi\left(\frac{1}{x}\right) + \psi\left(1-\frac{1}{x}\right) + 2\gamma\right\}$$
$$= -\frac{1}{x}\left\{2\psi\left(\frac{1}{x}\right) + \pi\cot\left(\frac{\pi}{x}\right) + 2\gamma\right\}$$
$$= -\frac{1}{x}\left\{2\psi\left(1+\frac{1}{x}\right) - 2x + \pi\cot\left(\frac{\pi}{x}\right) + 2\gamma\right\}.$$

Thus, using this last equality and Entry 9, we find that, for Re $x > 0$,

$$\varphi(2x) - \tfrac{1}{2}\varphi(x) - \frac{1}{x}\operatorname{Log} 2 = 1 + \frac{1}{x}\left\{\psi\left(1+\frac{1}{x}\right) - \psi\left(1+\frac{1}{2x}\right) - \operatorname{Log} 2\right\}$$
$$+ \frac{\pi}{2x}\left\{\cot\left(\frac{\pi}{x}\right) - \cot\left(\frac{\pi}{2x}\right)\right\}$$
$$= 1 - \int_0^1 \frac{u^x}{1+u^x}\,du - \frac{\pi}{2x}\csc\left(\frac{\pi}{x}\right).$$

In the notebooks, p. 92, in place of the right side of (13.1), Ramanujan has written

$$\text{“logarithmic part of } \sum_{k=0}^{\infty} \frac{(-1)^k}{kx+1}.\text{”}$$

Now, for Re $x > 0$,

$$1 - \int_0^1 \frac{u^x}{1+u^x}\,du = \sum_{k=0}^{\infty} \frac{(-1)^k}{kx+1},$$

but this provides only a partial explanation for Ramanujan's formulation.

The next two examples are easily established by expanding the integrands into geometric series.

Example 13(i). *If* $|x| < 1$ *and* Re $n > 0$, *then*

$$\int_0^x \frac{du}{1+u^n} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{nk+1}}{nk+1}.$$

Example 13(ii). *If* $|x| < 1$ *and* Re $n > 0$, *then*

$$\int_0^x \frac{du}{1-u^n} = \sum_{k=0}^{\infty} \frac{x^{nk+1}}{nk+1}.$$

The next two examples are trivial.

Example 13(iii). *If* $|x| < 1$ *and n is odd, then*

$$\int_0^x \frac{du}{1-u^n} = \int_0^x \frac{du}{1+(-u)^n}.$$

Example 13(iv). *If* $|x| < 1$ *and n is even, then*

$$\int_0^x \frac{du}{1-u^n} = \frac{1}{2}\int_0^x \frac{du}{1+u^{n/2}} + \frac{1}{2}\int_0^x \frac{du}{1-u^{n/2}}.$$

Example 13(v). *Let l and n be positive integers with* $l < n+1$. If n is even, then

(a)
$$\int \frac{x^{l-1}\,dx}{x^n-1} = \frac{1}{n}\operatorname{Log}(x-1) + \frac{(-1)^l}{n}\operatorname{Log}(x+1)$$
$$+\frac{1}{n}\sum_{r=1}^{\frac{1}{2}n-1} \cos\left(\frac{2rl\pi}{n}\right)\operatorname{Log}\left(x^2 - 2x\cos\left(\frac{2\pi r}{n}\right)+1\right)$$
$$-\frac{2}{n}\sum_{r=1}^{\frac{1}{2}n-1} \sin\left(\frac{2rl\pi}{n}\right)\tan^{-1}\left(\frac{x-\cos(2r\pi/n)}{\sin(2r\pi/n)}\right).$$

If n is odd, then

(b)
$$\int \frac{x^{l-1}\,dx}{x^n+1} = \frac{(-1)^{l-1}}{n}\operatorname{Log}(x+1)$$
$$-\frac{1}{n}\sum_{r=1}^{\frac{1}{2}(n-1)} \cos\left(\frac{(2r-1)l\pi}{n}\right)\operatorname{Log}\left(x^2 - 2x\cos\left(\frac{(2r-1)\pi}{n}\right)+1\right)$$
$$+\frac{2}{n}\sum_{r=1}^{\frac{1}{2}(n-1)} \sin\left(\frac{(2r-1)l\pi}{n}\right)\tan^{-1}\left(\frac{x-\cos((2r-1)\pi/n)}{\sin((2r-1)\pi/n)}\right).$$

Example 13(vi). *Let l and n be positive integers with $l < n+1$. If n is odd, then*

(a) $$\int \frac{x^{l-1}\,dx}{x^n - 1} = \frac{1}{n}\operatorname{Log}(x-1)$$
$$+\frac{1}{n}\sum_{r=1}^{\frac{1}{2}(n-1)} \cos\left(\frac{2rl\pi}{n}\right)\operatorname{Log}\left(x^2 - 2x\cos\left(\frac{2r\pi}{n}\right)+1\right)$$
$$-\frac{2}{n}\sum_{r=1}^{\frac{1}{2}(n-1)} \sin\left(\frac{2rl\pi}{n}\right)\tan^{-1}\left(\frac{x-\cos(2r\pi/n)}{\sin(2r\pi/n)}\right).$$

If n is even, then

(b) $$\int \frac{x^{l-1}\,dx}{x^n + 1}$$
$$= -\frac{1}{n}\sum_{r=1}^{\frac{1}{2}n} \cos\left(\frac{(2r-1)l\pi}{n}\right)\operatorname{Log}\left(x^2 - 2x\cos\left(\frac{(2r-1)\pi}{n}\right)+1\right)$$
$$+\frac{2}{n}\sum_{r=1}^{\frac{1}{2}n} \sin\left(\frac{(2r-1)l\pi}{n}\right)\tan^{-1}\left(\frac{x-\cos((2r-1)\pi/n)}{\sin((2r-1)\pi/n)}\right).$$

The integrals in Examples 13(v), (vi) may be routinely evaluated by expanding the integrands into partial fractions. Moreover, these four integrals may be found in the tables of Gradshteyn and Ryzhik [1, pp. 64, 65]. Ramanujan inadvertently omitted the latter summation sign in Example 13(vi), (b).

Entry 14. *Let*

$$A_n = \int_0^x \frac{du}{1+u^n}, \qquad n \geq 1.$$

Then

(i) $A_1 = \operatorname{Log}(1+x)$,

(ii) $A_2 = \tan^{-1} x$,

(iii) $$A_3 = \frac{1}{6}\operatorname{Log}\left(\frac{(1+x)^3}{1+x^3}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{2-x}\right),$$

(iv) $$A_4 = \frac{1}{4\sqrt{2}}\operatorname{Log}\left(\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}\right) + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x\sqrt{2}}{1-x^2}\right),$$

$$\text{(v)}\quad A_5 = \frac{1}{20}\operatorname{Log}\left(\frac{(1+x)^5}{1+x^5}\right) + \frac{1}{4\sqrt{5}}\operatorname{Log}\left(\frac{1+\frac{1}{2}x(\sqrt{5}-1)+x^2}{1-\frac{1}{2}x(\sqrt{5}+1)+x^2}\right)$$

$$+\frac{1}{10}\sqrt{10+2\sqrt{5}}\tan^{-1}\left(\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}-1)}\right)$$

$$+\frac{1}{10}\sqrt{10-2\sqrt{5}}\tan^{-1}\left(\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)}\right),$$

$$\text{(vi)}\quad A_6 = \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{6}\tan^{-1}x^3 + \frac{1}{4\sqrt{3}}\operatorname{Log}\left(\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}\right),$$

$$\text{(vii)}\quad A_8 = \frac{\sqrt{2+\sqrt{2}}}{16}\left\{\operatorname{Log}\left(\frac{1+x\sqrt{2+\sqrt{2}}+x^2}{1-x\sqrt{2+\sqrt{2}}+x^2}\right)\right.$$

$$\left.+2\tan^{-1}\left(\frac{x\sqrt{2+\sqrt{2}}}{1-x^2}\right)\right\}$$

$$+\frac{\sqrt{2-\sqrt{2}}}{16}\left\{\operatorname{Log}\left(\frac{1+x\sqrt{2-\sqrt{2}}+x^2}{1-x\sqrt{2-\sqrt{2}}+x^2}\right)\right.$$

$$\left.+2\tan^{-1}\left(\frac{x\sqrt{2-\sqrt{2}}}{1-x^2}\right)\right\},$$

and

$$\text{(viii)}\quad A_{10} = \tfrac{1}{5}\tan^{-1}x + \frac{\sqrt{6+2\sqrt{5}}}{20}\tan^{-1}\left(\frac{x\sqrt{6+2\sqrt{5}}}{2(1-x^2)}\right)$$

$$+\frac{\sqrt{6-2\sqrt{5}}}{20}\tan^{-1}\left(\frac{x\sqrt{6-2\sqrt{5}}}{2(1-x^2)}\right)$$

$$+\frac{\sqrt{10+2\sqrt{5}}}{40}\operatorname{Log}\left(\frac{1+\frac{1}{2}x\sqrt{10+2\sqrt{5}}+x^2}{1-\frac{1}{2}x\sqrt{10+2\sqrt{5}}+x^2}\right)$$

$$+\frac{\sqrt{10-2\sqrt{5}}}{40}\operatorname{Log}\left(\frac{1+\frac{1}{2}x\sqrt{10-2\sqrt{5}}+x^2}{1-\frac{1}{2}x\sqrt{10-2\sqrt{5}}+x^2}\right).$$

Proof. Parts (i) and (ii) are elementary.

To prove (iii) we put $n = 3$ and $l = 1$ in Example 13(v), (b) to get

$$\int_0^x \frac{du}{u^3+1} = \tfrac{1}{3}\operatorname{Log}(x+1) - \tfrac{1}{6}\operatorname{Log}(x^2-x+1)$$

$$+\frac{1}{\sqrt{3}}\left\{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\},$$

and the desired formula for A_3 follows upon simplification.

Before applying Example 13(vi), (b) to prove (iv), it is perhaps best to employ the fact

$$\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)-\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right)\equiv\tfrac{1}{2}\pi-\theta \pmod{\pi}. \quad (14.1)$$

Putting $n=4$ and $l=1$ in Example 13(vi), (b) then yields

$$A_4=-\frac{1}{4\sqrt{2}}\{\mathrm{Log}(x^2-x\sqrt{2}+1)-\mathrm{Log}(x^2+x\sqrt{2}+1)\}$$

$$+\frac{1}{2\sqrt{2}}\left\{\tan^{-1}\left(\frac{x}{\sqrt{2}-x}\right)+\tan^{-1}\left(\frac{x}{\sqrt{2}+x}\right)\right\}.$$

Part (iv) now readily follows.

The remaining calculations are similar to those above but are somewhat laborious. All computations are facilitated by using (14.1). To prove (v), put $n=5$ and $l=1$ in Example 13(v), (b). Parts (vi)–(viii) follow from putting $l=1$ and $n=6$, 8, and 10, respectively, in Example 13(vi), (b).

There are two discrepancies between our formulation of Entry 14 and that of Ramanujan, p. 94. In the second expression on the right side of (v), Ramanujan has written $\frac{1}{2}(\sqrt{5}-1)$ in the denominator instead of $\frac{1}{2}(\sqrt{5}+1)$. In the formula for A_{10}, Ramanujan has replaced the three terms involving the inverse tangent function by

$$\tfrac{1}{4}\tan^{-1}x-\frac{1}{20}\tan^{-1}x^5+\frac{1}{4\sqrt{5}}\tan^{-1}\left(\frac{(x-x^3)\sqrt{5}}{1-3x^2+x^4}\right).$$

Example 14.1. *We have*

(i) $$\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)2^{3k+1}}=\frac{\pi}{6\sqrt{3}}+\tfrac{1}{6}\mathrm{Log}\,3,$$

(ii) $$\sum_{k=0}^{\infty}\frac{(-1)^k(\sqrt{3}-1)^{3k+1}}{3k+1}=\frac{\pi}{4\sqrt{3}}+\tfrac{1}{3}\mathrm{Log}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right),$$

and

(iii) $$\sum_{k=0}^{\infty}\frac{(-1)^k(2-\sqrt{3})^{2k+1}}{4k+1}=\frac{(\sqrt{3}-1)\pi}{16}-\frac{(\sqrt{3}-1)}{4}\mathrm{Log}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right).$$

Proof. To establish (i), put $x=\frac{1}{2}$ and $n=3$ in Example 13(i) and use Entry 14(iii).

To prove (ii), set $x=\sqrt{3}-1$ and $n=3$ in Example 13(i) and employ Entry 14(iii).

Lastly, put $x=(\sqrt{3}-1)/\sqrt{2}$ and $n=4$ in Example 13(i) and employ Entry 14(iv). After multiplying both sides by x and observing that $x^2=2-\sqrt{3}$, we find that

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2-\sqrt{3})^{2k+1}}{4k+1}=\frac{\sqrt{3}-1}{\sqrt{2}}\int_0^{(\sqrt{3}-1)/\sqrt{2}}\frac{du}{1+u^4}$$

$$=\frac{\sqrt{3}-1}{8}\operatorname{Log}\left(\frac{1}{2-\sqrt{3}}\right)+\frac{(\sqrt{3}-1)\pi}{16},$$

from which (iii) follows.

On the right side of (iii) Ramanujan inadvertently omitted the factor $\sqrt{2}$ in the denominator.

Example 14.2. *With* $\varphi(x)$ *defined by* (11.1), *we have*

(i) $\varphi(2)=2\operatorname{Log}2,$

(ii) $\varphi(3)=\operatorname{Log}3,$

(iii) $\varphi(4)=\frac{3}{2}\operatorname{Log}2,$

(iv) $\varphi(6)=\frac{1}{2}\operatorname{Log}3+\frac{1}{3}\operatorname{Log}4,$

(v) $\varphi(5)=\frac{1}{2}\operatorname{Log}5+\dfrac{1}{\sqrt{5}}\operatorname{Log}\dfrac{\sqrt{5}+1}{2},$

(vi) $\varphi(8)=\operatorname{Log}2+\dfrac{\sqrt{2}}{4}\operatorname{Log}(\sqrt{2}+1),$

(vii) $\varphi(10)=\frac{2}{5}\operatorname{Log}2+\frac{1}{4}\operatorname{Log}5+\dfrac{3}{2\sqrt{5}}\operatorname{Log}\dfrac{\sqrt{5}+1}{2},$

(viii) $\varphi(12)=\dfrac{3+\sqrt{3}}{6}\operatorname{Log}2+\frac{1}{4}\operatorname{Log}3-\dfrac{1}{\sqrt{3}}\operatorname{Log}(\sqrt{3}-1),$

(ix) $$\varphi(16)=\frac{5}{8}\operatorname{Log}2+\frac{1}{4\sqrt{2}}\operatorname{Log}(\sqrt{2}+1)$$
$$+\frac{\sqrt{2+\sqrt{2}}}{16}\operatorname{Log}\left(\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}\right)$$
$$+\frac{\sqrt{2-\sqrt{2}}}{16}\operatorname{Log}\left(\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}\right),$$

and

$$\text{(x)}\quad \varphi(20)=\tfrac{3}{10}\operatorname{Log}2+\tfrac{1}{8}\operatorname{Log}5+\frac{3}{4\sqrt{5}}\operatorname{Log}\left(\frac{\sqrt{5}+1}{2}\right)$$
$$+\frac{\sqrt{10+2\sqrt{5}}}{40}\operatorname{Log}\left(\frac{4+\sqrt{10+2\sqrt{5}}}{4-\sqrt{10+2\sqrt{5}}}\right)$$
$$+\frac{\sqrt{10-2\sqrt{5}}}{40}\operatorname{Log}\left(\frac{4+\sqrt{10-2\sqrt{5}}}{4-\sqrt{10-2\sqrt{5}}}\right).$$

Proof. Parts (i)–(iv) have been proved in Chapter 2, Corollaries of Entries 1, 2, 4, and 5, and so we shall forego proofs here.

If n is a natural number, then by Entry 12,

$$\begin{aligned}\varphi(n)&=1+\int_0^1\frac{u^{n-2}(1-u)^2}{1-u^n}\,du\\&=1+\int_0^1\frac{1+u^{n-2}-2u^{n-1}}{1-u^n}\,du-\int_0^1\frac{1-u^n}{1-u^n}\,du\\&=\int_0^1\frac{1+u^{n-2}-2u^{n-1}}{1-u^n}\,du.\end{aligned}\tag{14.2}$$

Thus, in the sequel, we shall be making several applications of Examples 13(v), (a) and 13(vi), (a) with $l=1,\ n-1,\ n$. Note that when $l=n$, the sum involving inverse tangents is identically equal to 0. Also observe that the sum of the inverse tangent contributions of the terms with $l=1$ and $l=n-1$ to $\varphi(n)$ is also equal to 0.

We now prove (v). By (14.2) and Example 13(vi) with $n=5$, we find that

$$\begin{aligned}\varphi(5)&=-\frac{2}{5}\sum_{r=1}^{2}\left(\cos\left(\frac{2\pi r}{5}\right)-1\right)\operatorname{Log}\left(2-2\cos\left(\frac{2\pi r}{5}\right)\right)\\&=-\frac{2}{5}\left\{\frac{\sqrt{5}-5}{4}\left(\operatorname{Log}\sqrt{5}-\operatorname{Log}\frac{\sqrt{5}+1}{2}\right)\right.\\&\qquad\left.-\frac{\sqrt{5}+5}{4}\left(\operatorname{Log}\sqrt{5}+\operatorname{Log}\frac{\sqrt{5}+1}{2}\right)\right\},\end{aligned}$$

from which, upon simplification, part (v) follows.

Because the details for the remaining calculations are quite straightforward but rather time consuming, we shall omit further proofs. In all cases we use (14.2). For parts (vi)–(x), use Example 13(v), (a) with $n=8$, 10, 12, 16, and 20, respectively.

In Ramanujan's statement of (viii), p. 95, the term $(\sqrt{3}/6)\operatorname{Log}2$ does not appear. For a further study of the function $\varphi(x)$, see a paper of M. L. Robinson [1].

The content of the remainder of this chapter is much different from most of the foregoing material.

Ramanujan first defines a by the equality

$$\text{Log } a = \psi(x+1) \tag{15.1}$$

and regards a as a function of x, which we assume is real and positive. From Corollary 1 in Section 6, we note that Log $a \sim$ Log x as x tends to ∞.

Entry 15. *Let x and a be given as above, and suppose that n is any complex number. Then as a tends to* ∞,

$$\left(\frac{x+\frac{1}{2}}{a}\right)^{4n} \sim 1 - \frac{n}{6a^2} + \frac{10n^2+11n}{720a^4} - \frac{70n^3+231n^2+891n}{90720a^6} + \cdots. \tag{15.2}$$

Proof. Let $z > 0$. By Corollary 1 in Section 6, as z tends to ∞,

$$\psi(z+\tfrac{1}{2}) \sim \text{Log } z + \sum_{k=1}^{\infty} \frac{B_{2k}(1-2^{1-2k})}{2kz^{2k}}.$$

Putting $z = x + \frac{1}{2}$ and using (15.1), we find that

$$\begin{aligned}\text{Log}\left(\frac{z}{a}\right) &\sim -\sum_{k=1}^{\infty} \frac{B_{2k}(1-2^{1-2k})}{2kz^{2k}} \\ &= -\frac{1}{24z^2} + \frac{7}{960z^4} - \frac{31}{8064z^6} + \cdots.\end{aligned}$$

Hence,

$$\begin{aligned}\left(\frac{z}{a}\right)^{24} &\sim \exp\left(-\frac{1}{z^2} + \frac{7}{40z^4} - \frac{31}{336z^6} + \cdots\right) \\ &= 1 - \frac{1}{z^2} + \frac{27}{40z^4} - \frac{243}{560z^6} + \cdots.\end{aligned}$$

To find an asymptotic series in descending powers of a, we employ the method of successive approximations.

The first approximation is

$$\left(\frac{z}{a}\right)^{24} = 1 - \frac{1}{a^2}.$$

Thus, $z^{-2} = a^{-2}(1-a^{-2})^{-1/12}$, and so the second approximation is

$$\left(\frac{z}{a}\right)^{24} = 1 - \frac{1}{a^2}\left(1 + \frac{1}{12a^2}\right) + \frac{27}{40a^4} = 1 - \frac{1}{a^2} + \frac{71}{120a^4}.$$

Thus, $z^{-2} = a^{-2}(1 - 1/a^2 + 71/120a^4)^{-1/12}$, and so the third approximation is

$$\left(\frac{z}{a}\right)^{24} = 1 - \frac{1}{a^2}\left(1 + \frac{1}{12a^2} - \frac{71}{1440a^4} + \frac{13}{288a^4}\right)$$

$$+ \frac{27}{40a^4}\left(1 + \frac{1}{6a^2}\right) - \frac{243}{560a^6}$$

$$= 1 - \frac{1}{a^2} + \frac{71}{120a^4} - \frac{533}{1680a^6}.$$

Raising each side to the $n/6$ power, we deduce that

$$\left(\frac{z}{a}\right)^{4n} - 1 = \frac{n}{6a^2} + \frac{71n}{720a^4} - \frac{533n}{10080a^6}$$

$$+ \frac{n}{12}\left(\frac{n}{6} - 1\right)\left(-\frac{1}{a^2} + \frac{71}{120a^4}\right)^2$$

$$+ \frac{n}{36}\left(\frac{n}{6} - 1\right)\left(\frac{n}{6} - 2\right)\left(-\frac{1}{a^2}\right)^3 + \cdots.$$

Upon simplification, we obtain the proposed first four terms of the asymptotic expansion (15.2).

Corollary. "$\Gamma(x+1)$ *is minimum when* $x = 6/13$ *very nearly.*"

We have quoted Ramanujan's formulation of his corollary, p. 95. From (4.2), it is easy to see that $\Gamma(x+1)$ has exactly one minimum for positive values of x, and from (15.1), that minimum x_0 occurs when $a = 1$. Approximations for x_0 can be obtained from (15.2). Taking just two terms, we find that x_0 is approximately 11/24, and taking three terms, we find that x_0 is approximately equal to 889/1920. It is interesting to read Ramanujan's argument, "$x = \frac{1}{2} - \frac{1}{24} + \&c$ or $x = 1/(2 + 1/6)$ very nearly." These values might be compared with the actual value of x_0 (see Whittaker and Watson's text [1, p. 253]);

$$x_0 = 0.4616321\ldots,$$

$$11/24 = 0.4583333\ldots,$$

$$889/1920 = 0.4630208\ldots,$$

$$6/13 = 0.4615384\ldots.$$

Thus, Ramanujan's "interpolated" value is better than either of the approximations 11/24 or 889/1920.

Entry 16. *Let* $A_k = \frac{1}{2}(3^k - 1)$, $k \geq 0$. *Then*

$$\gamma = \operatorname{Log} 2 - 2 \sum_{k=1}^{\infty} k \sum_{j=A_{k-1}+1}^{A_k} \frac{1}{(3j)^3 - 3j}, \tag{16.1}$$

where γ *denotes Euler's constant.*

In Ramanujan's formulation of Entry 16, the words "the last term ...", p. 95, should be replaced by "the first term ...." Ramanujan [3], [15, p. 325], submitted (16.1) as a problem to the *Journal of the Indian Mathematical Society*. Evidently, a solution was never published.

Proof. From the corollary to Entry 6 of Chapter 2,

$$\sum_{k=1}^{A_n} \frac{1}{k} = n + 2 \sum_{k=1}^{n-1} (n-k) \sum_{j=A_{k-1}+1}^{A_k} \frac{1}{(3j)^3 - 3j}, \qquad n \geq 1.$$

Rearranging this equality and then taking the limit as n tends to ∞, we find that

$$\begin{aligned}
-2 \sum_{k=1}^{\infty} k & \sum_{j=A_{k-1}+1}^{A_k} \frac{1}{(3j)^3 - 3j} \\
&= \lim_{n \to \infty} \left\{ \sum_{k=1}^{A_n} \frac{1}{k} - \operatorname{Log}\left(\frac{3^n}{2}\right) + \operatorname{Log}\left(\frac{3^n}{2}\right) \right. \\
&\qquad \left. - n\left(1 + 2 \sum_{k=1}^{n-1} \sum_{j=A_{k-1}+1}^{A_k} \frac{1}{(3j)^3 - 3j}\right) \right\} \\
&= \gamma - \operatorname{Log} 2 + \lim_{n \to \infty} n \left\{ \operatorname{Log} 3 - 1 - 2 \sum_{k=1}^{n-1} \sum_{j=A_{k-1}+1}^{A_k} \frac{1}{(3j)^3 - 3j} \right\} \\
&= \gamma - \operatorname{Log} 2 + \lim_{n \to \infty} n \sum_{k=A_{n-1}+1}^{\infty} \frac{1}{(3j)^3 - 3j} \\
&= \gamma - \operatorname{Log} 2,
\end{aligned}$$

where in the penultimate equality we employed Example 14.2(ii). This completes the proof.

Entry 17(i). *Let*

$$\varphi(x) = \sum_{k=1}^{x} \frac{\operatorname{Log} k}{k}, \tag{17.1}$$

$$c_1 = \lim_{n \to \infty} \{\varphi(n) - \tfrac{1}{2} \operatorname{Log}^2 n\}, \tag{17.2}$$

and

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{17.3}$$

Then as x tends to ∞,

$$\varphi(x) - \psi(x+1)\operatorname{Log} x \sim -\tfrac{1}{2}\operatorname{Log}^2 x + c_1 + \sum_{k=1}^{\infty} \frac{B_{2k}H_{2k-1}}{2kx^{2k}},$$

where B_n denotes the nth *Bernoulli number.*

Proof. Apply the Euler–Maclaurin formula (I5) with $f(t) = (\operatorname{Log} t)/t$. Then as x tends to ∞,

$$\varphi(x) \sim \frac{\operatorname{Log}^2 x}{2} + \frac{\operatorname{Log} x}{2x} + c + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x), \tag{17.4}$$

where c is a constant. It is easy to see from (17.4) that $c = c_1$. Now if $f_1(x) = \operatorname{Log} x$ and $f_2(x) = 1/x$, then

$$f_1^{(j)}(x) = \frac{(-1)^{j-1}(j-1)!}{x^j} \quad \text{and} \quad f_2^{(j)}(x) = \frac{(-1)^j j!}{x^{j+1}},$$

where $j \geq 1$. Then a routine calculation with the use of Leibniz's rule shows that

$$f^{(n)}(x) = \frac{(-1)^n n! \operatorname{Log} x}{x^{n+1}} + \frac{(-1)^{n-1} n!\, H_n}{x^{n+1}}, \qquad n \geq 1.$$

Thus, from (17.4),

$$\varphi(x) \sim \frac{\operatorname{Log}^2 x}{2} + \frac{\operatorname{Log} x}{2x} + c_1 - \operatorname{Log} x \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}} + \sum_{k=1}^{\infty} \frac{B_{2k}H_{2k-1}}{2kx^{2k}}, \tag{17.5}$$

as x tends to ∞. From Stirling's formula found in Chapter 7, Entry 23, as x tends to ∞,

$$\psi(x+1) \sim \operatorname{Log} x + \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}}. \tag{17.6}$$

Combining (17.5) and (17.6), we complete the proof.

Ramanujan gives the value $c_1 = -0.072815845483680$, which agrees with a calculation of Liang and Todd [1] except for the last two decimal places which should be 77 instead of 80.

Corollary. *Let $\varphi(x)$ and c_1 be defined by* (17.1) *and* (17.2), *respectively. Then*

$$\lim_{x\to\infty} \{\varphi(x) - \tfrac{1}{2}\psi^2(x+1)\} = c_1.$$

Proof. This result follows readily from (17.5) and (17.6).

Entry 17(ii). *If x is real, then*

$$\varphi(x) = \sum_{k=1}^{\infty} \left\{ \frac{\operatorname{Log} k}{k} - \frac{\operatorname{Log}(k+x)}{k+x} \right\}. \tag{17.7}$$

Now if x is a positive integer, Entry 17(ii) is trivial. For nonintegral x, Ramanujan is actually defining $\varphi(x)$ by (17.7). This device for extending the definition of a sum is frequently used by Ramanujan (e.g., see (3.1)).

Corollary. *If* $\varphi(x)$ *is defined by* (17.7), *then*

$$\sum_{k=0}^{\infty} \frac{(-1)^k \operatorname{Log}(2k+1)}{2k+1} = \tfrac{1}{4}\{\varphi(-\tfrac{1}{4}) - \varphi(-\tfrac{3}{4})\} + \tfrac{1}{2}\pi \operatorname{Log} 2.$$

Proof. By (17.7),

$$\begin{aligned}\tfrac{1}{4}\{\varphi(-\tfrac{1}{4}) - \varphi(-\tfrac{3}{4})\} &= \sum_{k=1}^{\infty} \left\{\frac{\operatorname{Log}(k-\tfrac{3}{4})}{4k-3} - \frac{\operatorname{Log}(k-\tfrac{1}{4})}{4k-1}\right\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \operatorname{Log}(2k+1)}{2k+1} - \operatorname{Log} 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1},\end{aligned}$$

and the desired result follows from Leibniz's series for $\pi/4$.

Entry 17(iii). *If* $\varphi(x)$ *is defined by* (17.7) *and* n *is a positive integer, then*

$$n\varphi(x) - \sum_{k=0}^{n-1} \varphi\left(\frac{x-k}{n}\right) = n \operatorname{Log} n\, \psi(x+1) - \tfrac{1}{2}\, n \operatorname{Log}^2 n.$$

Proof. By (17.7),

$$\begin{aligned} n\varphi(x) &- \sum_{k=0}^{n-1} \varphi\left(\frac{x-k}{n}\right) \\ &= n \lim_{m\to\infty} \sum_{j=1}^{mn} \left\{\frac{\operatorname{Log} j}{j} - \frac{\operatorname{Log}(j+x)}{j+x}\right\} \\ &\quad - \lim_{m\to\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{m} \left\{\frac{\operatorname{Log} j}{j} - \frac{n \operatorname{Log}(nj-k+x) - n \operatorname{Log} n}{nj-k+x}\right\} \\ &= \lim_{m\to\infty} \left\{n \sum_{j=m+1}^{mn} \frac{\operatorname{Log} j}{j} - n \operatorname{Log} n \sum_{j=1}^{mn} \frac{1}{j+x}\right\} \\ &= \lim_{m\to\infty} \left\{\tfrac{1}{2} n\, (\operatorname{Log}^2(mn) - \operatorname{Log}^2 m) - n \operatorname{Log} n \sum_{j=1}^{mn} \frac{1}{j}\right. \\ &\quad \left. + n \operatorname{Log} n \sum_{j=1}^{mn} \left(\frac{1}{j} - \frac{1}{j+x}\right)\right\} \\ &= \lim_{m\to\infty} \{n \operatorname{Log} n \operatorname{Log} m + \tfrac{1}{2} n \operatorname{Log}^2 n - n \operatorname{Log} n\, (\operatorname{Log} m + \operatorname{Log} n + \gamma)\} \\ &\quad + n \operatorname{Log} n\, (\psi(x+1) + \gamma) \\ &= -\tfrac{1}{2} n \operatorname{Log}^2 n + n \operatorname{Log} n\, \psi(x+1), \end{aligned}$$

where in the penultimate equality we used (4.2).

Corollary. *If* $\varphi(x)$ *is defined by* (17.7), *then*

$$\sum_{k=1}^{n-1} \varphi\left(-\frac{k}{n}\right) = \gamma n \operatorname{Log} n + \tfrac{1}{2} n \operatorname{Log}^2 n. \tag{17.8}$$

Proof. Set $x = 0$ in Entry 17(iii) and use the fact that $\psi(1) = -\gamma$, as can be seen from (4.2).

Example 1. *We have*

$$\prod_{k=1}^{\infty} \frac{(2k-1)^{1/(2k-1)}}{(2k)^{1/(2k)}} = 2^{\frac{1}{2}\operatorname{Log} 2 - \gamma}.$$

Proof. Let

$$P_n = \prod_{k=1}^{n} \frac{(2k-1)^{1/(2k-1)}}{(2k)^{1/(2k)}}, \qquad n \geq 1.$$

Then

$$\begin{aligned} \operatorname{Log} P_n &= \sum_{k=1}^{2n} \frac{(-1)^{k-1} \operatorname{Log} k}{k} \\ &= \sum_{k=1}^{2n} \frac{\operatorname{Log} k}{k} - \sum_{k=1}^{n} \frac{\operatorname{Log}(2k)}{k} \\ &= \tfrac{1}{2} \operatorname{Log}^2 (2n) + c_1 - \tfrac{1}{2} \operatorname{Log}^2 n - c_1 - \operatorname{Log} 2 \,(\operatorname{Log} n + \gamma) + o(1) \\ &= \tfrac{1}{2} \operatorname{Log}^2 2 - \gamma \operatorname{Log} 2 + o(1), \end{aligned}$$

as n tends to ∞, where c_1 is defined by (17.2). Exponentiating and letting n tend to ∞, we complete the proof.

Example 2. $\varphi(-\frac{1}{2}) = 2\gamma \operatorname{Log} 2 + \operatorname{Log}^2 2.$

Example 3. $\varphi(-\frac{1}{3}) + \varphi(-\frac{2}{3}) = 3\gamma \operatorname{Log} 3 + \frac{3}{2} \operatorname{Log}^2 3.$

Example 4. $\varphi(-\frac{1}{4}) + \varphi(-\frac{3}{4}) = 6\gamma \operatorname{Log} 2 + 7 \operatorname{Log}^2 2.$

Example 5. $\varphi(-\frac{1}{6}) + \varphi(-\frac{5}{6}) = 3\gamma \operatorname{Log} 3 + 4\gamma \operatorname{Log} 2 + \frac{3}{2} \operatorname{Log}^2(12) - \operatorname{Log}^2 4.$

Examples 2–5 follow from setting $n = 2, 3, 4$, and 6, respectively, in (17.8).

Entry 17(iv). *If* $0 < x < 1$, *then*

$$\tfrac{1}{2}\pi \left\{ \operatorname{Log} \frac{\Gamma(x)}{\Gamma(1-x)} + (\gamma + \operatorname{Log}(2\pi))(2x-1) \right\} = \sum_{k=1}^{\infty} \frac{\operatorname{Log} k}{k} \sin(2\pi k x).$$

Proof. By a result of Kummer [1] (see also Whittaker and Watson [1, p. 250]), if $0 < x < 1$,

$$\begin{aligned} \operatorname{Log} \Gamma(x) = \tfrac{1}{2} \operatorname{Log} \pi - \tfrac{1}{2} \operatorname{Log}(\sin(\pi x)) \\ + \sum_{k=1}^{\infty} \frac{1}{k\pi} (\gamma + \operatorname{Log}(2\pi k)) \sin(2\pi k x). \end{aligned} \tag{17.9}$$

Replacing x by $1-x$, we find that

$$\begin{aligned}\operatorname{Log}\Gamma(1-x) &= \tfrac{1}{2}\operatorname{Log}\pi - \tfrac{1}{2}\operatorname{Log}(\sin(\pi x)) \\ &\quad - \sum_{k=1}^{\infty} \frac{1}{k\pi}(\gamma + \operatorname{Log}(2\pi k))\sin(2\pi k x). \qquad (17.10)\end{aligned}$$

Subtracting (17.10) from (17.9) and using the fact that

$$\sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k\pi} = \tfrac{1}{2} - x, \qquad 0 < x < 1,$$

we easily complete the proof.

Entry 17(v). *Let* $\varphi(x)$ *be defined by* (17.7). *If* $0 < x < 1$, *then*

$$\varphi(x-1) - \varphi(-x) = (\gamma + \operatorname{Log}(2\pi))\pi\cot(\pi x) + 2\pi \sum_{k=1}^{\infty} \sin(2\pi k x)\operatorname{Log} k. \qquad (17.11)$$

Of course, Entry 17(v) is meaningless because the series on the right side diverges for $0 < x < 1$. Entry 17(v) is intended to be an analogue of Entry 17(iv). In the midst of his formula, after $\cot(\pi x)$, Ramanujan inserts a parenthetical remark "for the same limits," the meaning of which we are unable to discern. After his formula, Ramanujan informs us to note that

$$\sum_{k=1}^{\infty} \sin(2\pi k x) = \tfrac{1}{2}\cot(\pi x), \qquad (17.12)$$

which also is devoid of meaning. (Formula (17.12) may be formally established by differentiating the well-known equality

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{k} = -\operatorname{Log}(2\sin(\pi x)), \qquad 0 < x < 1, \qquad (17.13)$$

with respect to x.)

Nonetheless, we are able to formally derive (17.11), and we do so now to show how Ramanujan must have thought. For $\sigma > 0$ and $0 < x < 1$, define

$$G(s) = \zeta(s, x) - \zeta(s, 1-x) = \sum_{k=0}^{\infty} \{(k+x)^{-s} - (k+1-x)^{-s}\},$$

where $\zeta(s, a)$ denotes the Hurwitz zeta-function. Thus, for $\sigma > 0$,

$$G'(s) = \sum_{k=0}^{\infty} \left\{\frac{\operatorname{Log}(k+1-x)}{(k+1-x)^s} - \frac{\operatorname{Log}(k+x)}{(k+x)^s}\right\},$$

and by (17.7),

$$G'(1) = \varphi(x-1) - \varphi(-x). \qquad (17.14)$$

Recall Hurwitz's formula, found in Titchmarsh's treatise [3, p. 37],

$$\zeta(s, a) = 2\Gamma(1-s)\left\{\sin(\tfrac{1}{2}\pi s)\sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{(2\pi k)^{1-s}} + \cos(\tfrac{1}{2}\pi s)\sum_{k=1}^{\infty} \frac{\sin(2\pi k a)}{(2\pi k)^{1-s}}\right\}, \qquad (17.15)$$

where $\sigma < 1$ and $0 < a < 1$. It follows that, for $\sigma < 1$ and $0 < x < 1$,

$$G(s) = 4\Gamma(1-s)\cos(\tfrac{1}{2}\pi s)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{(2\pi k)^{1-s}}.$$

Hence, for $\sigma < 1$,

$$G'(s) = \{-4\Gamma'(1-s)\cos(\tfrac{1}{2}\pi s) - 2\pi\Gamma(1-s)\sin(\tfrac{1}{2}\pi s)\}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{(2\pi k)^{1-s}}$$

$$+ 4\Gamma(1-s)\cos(\tfrac{1}{2}\pi s)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)\operatorname{Log}(2\pi k)}{(2\pi k)^{1-s}}. \qquad (17.16)$$

Using the expansion (Abramowitz and Stegun [1, p. 256]),

$$\Gamma(1-s) = \frac{1}{1-s} - \gamma + \cdots,$$

employing (17.14), and letting s tend to $1-$ in (17.16). we find formally that

$$\varphi(x-1) - \varphi(-x) = 2\pi\gamma\sum_{k=1}^{\infty}\sin(2\pi kx) + 2\pi\sum_{k=1}^{\infty}\sin(2\pi kx)\operatorname{Log}(2\pi k).$$

If we now apply (17.12), we formally deduce (17.11).

In Example 1, Ramanujan asks us to "Find $\varphi(-\frac{1}{2})$, $\varphi(-\frac{2}{3})$, $\varphi(-\frac{3}{4})$ and $\varphi(-\frac{5}{6})$." Except for the fact that $\varphi(-\frac{1}{2})$ has been previously determined in Example 2 following Entry 17(iii), it is significant that Ramanujan uncharacteristically does not record the values of $\varphi(-\frac{2}{3})$, $\varphi(-\frac{3}{4})$, and $\varphi(-\frac{5}{6})$. If Ramanujan really had a bona fide formula for $\varphi(x-1) - \varphi(-x)$, then this, in conjunction with Examples 3, 4, and 5 following Entry 17(iii), could be used to determine $\varphi(-\frac{2}{3})$, $\varphi(-\frac{3}{4})$, and $\varphi(-\frac{5}{6})$.

Example 2. *We have*

$$\sum_{k=0}^{\infty}\frac{(-1)^k\operatorname{Log}(2k+1)}{2k+1} = \tfrac{1}{4}\pi\operatorname{Log}\pi - \pi\operatorname{Log}\Gamma(\tfrac{3}{4}) - \tfrac{1}{4}\pi\gamma.$$

Proof. Putting $x = \frac{3}{4}$ in Kummer's formula (17.9), we find that

$$\operatorname{Log}\Gamma(\tfrac{3}{4}) = \tfrac{1}{2}\operatorname{Log}\pi + \tfrac{1}{4}\operatorname{Log}2 + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\{\gamma + \operatorname{Log}(2\pi) + \operatorname{Log}(2k+1)\}.$$

Using Leibniz's series for $\pi/4$ and simplifying, we reach the desired result.

Example 3. *We have*

$$\frac{\left\{\prod_{k=1}^{\infty}\dfrac{(2k-1)^{1/(2k-1)}}{(2k)^{1/(2k)}}\right\}^{1/\operatorname{Log}2}}{\left\{\prod_{k=1}^{\infty}\dfrac{(4k-3)^{1/(4k-3)}}{(4k-1)^{1/(4k-1)}}\right\}^{4/\pi}} = \frac{\sqrt{2}}{\pi}\Gamma^4(\tfrac{3}{4}).$$

Proof. From Example 1 following Entry 17(iii), we have

$$\left\{\prod_{k=1}^{\infty} \frac{(2k-1)^{1/(2k-1)}}{(2k)^{1/(2k)}}\right\}^{1/\text{Log } 2} = \sqrt{2}e^{-\gamma}. \tag{17.17}$$

Also, by Example 2 above,

$$\left\{\prod_{k=1}^{\infty} \frac{(4k-3)^{1/(4k-3)}}{(4k-1)^{1/(4k-1)}}\right\}^{4/\pi}$$

$$= \exp \frac{4}{\pi}\left(\tfrac{1}{4}\pi \text{ Log } \pi - \pi \text{ Log } \Gamma(\tfrac{3}{4}) - \tfrac{1}{4}\pi\gamma\right) = \pi\Gamma^{-4}(\tfrac{3}{4})e^{-\gamma}. \tag{17.18}$$

Combining (17.17) and (17.18), we complete the proof.

Entry 18(i). *Let c_1 and H_n be defined by* (17.2) *and* (17.3), *respectively. Define*

$$\varphi(x) = \sum_{k=1}^{x} \text{Log}^2 k \tag{18.1}$$

and

$$C = \tfrac{1}{2}\gamma^2 + c_1 - \tfrac{1}{24}\pi^2 - \tfrac{1}{2} \text{ Log}^2(2\pi). \tag{18.2}$$

Then as x tends to ∞,

$$\varphi(x) - 2 \text{ Log } x \text{ Log}\left(\frac{\Gamma(x+1)}{\sqrt{2\pi}}\right) \sim -(x+\tfrac{1}{2})\text{Log}^2 x + 2x + C$$

$$- 2\sum_{k=2}^{\infty} \frac{B_{2k}H_{2k-2}}{(2k-1)2kx^{2k-1}}.$$

Proof. We shall apply the Euler–Maclaurin formula (I5) with $f(t) = \text{Log}^2 t$. Two integrations by parts yield

$$\int_1^x \text{Log}^2 t \, dt = x \text{ Log}^2 x - 2x \text{ Log } x + 2(x-1),$$

and a straightforward application of Leibniz's formula shows that

$$f^{(2k-1)}(x) = 2(2k-2)! \, x^{1-2k}(\text{Log } x - H_{2k-2}), \qquad k \geq 1.$$

Thus, by (I5), as x tends to ∞,

$$\varphi(x) \sim x \text{ Log}^2 x - 2x \text{ Log } x + 2x + \tfrac{1}{2} \text{ Log}^2 x + c$$

$$+ 2 \text{ Log } x \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k-1)(2k)x^{2k-1}} - 2\sum_{k=1}^{\infty} \frac{B_{2k}H_{2k-2}}{(2k-1)(2k)x^{2k-1}}, \tag{18.3}$$

where c is some constant. Now by Stirling's formula, given in Chapter 7, Entry 23, as x tends to ∞,

$$\operatorname{Log}\Gamma(x+1) \sim (x+\tfrac{1}{2})\operatorname{Log} x - x + \tfrac{1}{2}\operatorname{Log}(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k-1)(2k)x^{2k-1}}. \tag{18.4}$$

Taking (18.3) and (18.4) together, we find that

$$\varphi(x) - 2\operatorname{Log} x \operatorname{Log}\left(\frac{\Gamma(x+1)}{\sqrt{2\pi}}\right) \sim -(x+\tfrac{1}{2})\operatorname{Log}^2 x + 2x + c - 2\sum_{k=1}^{\infty} \frac{B_{2k}H_{2k-2}}{(2k-1)(2k)x^{2k-1}}.$$

It remains to show that $c = C$, where C is given by (18.2).

First observe that, by (18.3),

$$c = \lim_{x\to\infty}\left\{\sum_{k=1}^{x} \operatorname{Log}^2 k - x\operatorname{Log}^2 x + 2x\operatorname{Log} x - 2x - \tfrac{1}{2}\operatorname{Log}^2 x\right\}. \tag{18.5}$$

Secondly, we show that $c = \zeta''(0)$, where ζ denotes the Riemann zeta-function. From Titchmarsh's treatise [3, pp. 14, 15],

$$\zeta(s) = s\int_1^{\infty} \frac{[x]-x+\frac{1}{2}}{x^{s+1}}\,dx + \frac{1}{s-1} + \frac{1}{2}, \qquad \sigma > -1. \tag{18.6}$$

Upon two differentiations and setting $s = 0$, we find that

$$\begin{aligned}
\zeta''(0) &= -2\int_1^{\infty} \frac{[x]-x+\frac{1}{2}}{x^{s+1}}\operatorname{Log} x\,dx - 2 \\
&= \lim_{n\to\infty}\left(-2\sum_{k=1}^{n-1} k\int_k^{k+1} \frac{\operatorname{Log} x}{x}\,dx + 2\int_1^n \operatorname{Log} x\,dx - \int_1^n \frac{\operatorname{Log} x}{x}\,dx\right) - 2 \\
&= \lim_{n\to\infty}\left(-\sum_{k=1}^{n-1} k\{\operatorname{Log}^2(k+1) - \operatorname{Log}^2 k\} + 2n\operatorname{Log} n - 2n - \tfrac{1}{2}\operatorname{Log}^2 n\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \operatorname{Log}^2 k - n\operatorname{Log}^2 n + 2n\operatorname{Log} n - 2n - \tfrac{1}{2}\operatorname{Log}^2 n\right).
\end{aligned} \tag{18.7}$$

Comparing (18.7) with (18.5), we find that $c = \zeta''(0)$.

We next calculate $\zeta''(0)$ by turning to the functional equation of $\zeta(s)$,

$$\zeta(s) = 2(2\pi)^{s-1} \sin(\tfrac{1}{2}\pi s)\Gamma(1-s)\zeta(1-s), \tag{18.8}$$

given by Entry 4 or (4.2) in Chapter 7. We expand both sides of (18.8) in powers of s. The values for $\zeta(0)$ and $\zeta'(0)$ may be found in Titchmarsh's book [1, pp. 19, 20], the expansion of $\zeta(1-s)$ about $s=0$ is determined by Ramanujan in Chapter 7, Entry 13, and the values $\Gamma'(1) = -\gamma$ and $\Gamma''(1) = \gamma^2 + \pi^2/6$ can be determined from (4.2). Accordingly, we find that

$$\begin{aligned} -\tfrac{1}{2} - \tfrac{1}{2}\operatorname{Log}(2\pi)s + \tfrac{1}{2}\zeta''(0)s^2 + \cdots \\ = (1 + \operatorname{Log}(2\pi)s + \tfrac{1}{2}\operatorname{Log}^2(2\pi)s^2 + \cdots)\left(\frac{s}{2} - \frac{\pi^2 s^3}{48} + \cdots\right) \\ \cdot\left(1 + \gamma s + \tfrac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right)s^2 + \cdots\right)\left(-\frac{1}{s} + \gamma + c_1 s + \cdots\right). \end{aligned}$$

Equating coefficients of s^2 on both sides, we deduce that

$$\begin{aligned} \tfrac{1}{2}\zeta''(0) &= \frac{\pi^2}{48} + \frac{c_1}{2} + \frac{\gamma^2}{2} + \frac{\gamma \operatorname{Log}(2\pi)}{2} - \frac{1}{4}\left(\gamma^2 + \frac{\pi^2}{6}\right) - \frac{\gamma \operatorname{Log}(2\pi)}{2} - \frac{\operatorname{Log}^2(2\pi)}{4} \\ &= -\frac{\pi^2}{48} + \frac{c_1}{2} + \frac{\gamma^2}{4} - \frac{\operatorname{Log}^2(2\pi)}{4}. \end{aligned}$$

Thus, we have shown that $c = \zeta''(0) = C$ and so have completed the proof of Entry 18(i).

In order to give meaning to Entry 18(ii), we shall need to extend the definition (18.1) of $\varphi(x)$ to the set of all real numbers. Unfortunately, Ramanujan does not divulge his more general definition of $\varphi(x)$. Clearly, an exact analogue of (17.7) is impossible. We shall indicate two ways of defining $\varphi(x)$ and then show that they are equivalent. First, we follow Nörlund [2, Chapters 3, 4], and define $\varphi(x)$ to be the "Hauptlösung" of the difference equation

$$\varphi(x) - \varphi(x-1) = \operatorname{Log}^2 x. \tag{18.9}$$

Thus, in view of (18.3), we define $\varphi(x)$ for $x > -1$ by

$$\begin{aligned} \varphi(x) = \lim_{n\to\infty} \Bigg\{ &(x+n)\operatorname{Log}^2(x+n) - 2(x+n)\operatorname{Log}(x+n) \\ &+ 2(x+n) + \tfrac{1}{2}\operatorname{Log}^2(x+n) - \sum_{k=1}^{n} \operatorname{Log}^2(x+k) - n\operatorname{Log}^2 n \\ &+ 2n\operatorname{Log} n - 2n - \tfrac{1}{2}\operatorname{Log}^2 n + \sum_{k=1}^{n} \operatorname{Log}^2 k \Bigg\} \end{aligned}$$

$$= \lim_{n\to\infty} \left\{(x+n)\left(\operatorname{Log} n + \frac{x}{n} + O\left(\frac{1}{n^2}\right)\right)^2 \right.$$

$$- 2(x+n)\left(\operatorname{Log} n + \frac{x}{n} + O\left(\frac{1}{n^2}\right)\right) + 2x$$

$$+ \tfrac{1}{2}\left(\operatorname{Log} n + O\left(\frac{1}{n}\right)\right)^2 - n \operatorname{Log}^2 n + 2n \operatorname{Log} n - \tfrac{1}{2}\operatorname{Log}^2 n$$

$$\left. + \sum_{k=1}^{n} (\operatorname{Log}^2 k - \operatorname{Log}^2(x+k))\right\}$$

$$= \lim_{n\to\infty} \left\{x \operatorname{Log}^2 n + \sum_{k=1}^{n} (\operatorname{Log}^2 k - \operatorname{Log}^2(x+k))\right\}. \tag{18.10}$$

It is easy to show, from the last equality, that $\varphi(x)$, indeed, is a solution of (18.9).

We now motivate our second definition of $\varphi(x)$. We first extend the usual definition of $\zeta(s, x)$ by setting

$$\zeta(s, x) = \sum_{k=0}^{\infty} (k+x)^{-s}, \qquad \sigma > 1, \tag{18.11}$$

for *all real* x. By a formula of Lerch [2],

$$\operatorname{Log} \Gamma(x) = \zeta'(0, x) - \zeta'(0). \tag{18.12}$$

(A simple proof of Lerch's formula has been given by Berndt [4].) With this in mind, define

$$\varphi(x) = -\zeta''(0, x+1) + \zeta''(0). \tag{18.13}$$

A brief calculation shows that, for $\sigma > 1$,

$$\zeta''(s, x) - \zeta''(s, x+1) = x^{-s} \operatorname{Log}^2 x. \tag{18.14}$$

By analytic continuation, (18.14) is valid for all complex values of s. It follows from (18.13) and (18.14) that (18.9) is satisfied. We next show that (18.10) and (18.13) are in agreement.

By a straightforward application of the Euler–Maclaurin formula (I3) to (18.11), we find that, for $\sigma > 1$ and $x > 0$,

$$\zeta(s, x) = \frac{x^{1-s}}{s-1} + \frac{x^{-s}}{2} - s\int_0^\infty \frac{t - [t] - \frac{1}{2}}{(t+x)^{s+1}}\, dt.$$

By analytic continuation, this formula holds for $\sigma > -1$. Differentiating twice and setting $s = 0$, we find that

$$\zeta''(0, x) = -x \operatorname{Log}^2 x + 2x \operatorname{Log} x - 2x + \tfrac{1}{2}\operatorname{Log}^2 x$$

$$+ 2\int_0^\infty \frac{t - [t] - \frac{1}{2}}{t+x} \operatorname{Log}(t+x)\, dt. \tag{18.15}$$

Now,

$$\int_0^\infty \frac{t-[t]-\frac{1}{2}}{t+x} \operatorname{Log}(t+x)dt$$

$$= \lim_{n\to\infty} \left(\int_0^n \frac{t+x}{t+x} \operatorname{Log}(t+x)\, dt - \sum_{k=1}^{n-1} k \int_k^{k+1} \frac{\operatorname{Log}(t+x)}{t+x} dt \right.$$

$$\left. - (\tfrac{1}{2}+x) \int_0^n \frac{\operatorname{Log}(t+x)}{t+x} dt \right)$$

$$= \lim_{n\to\infty} \left((n+x) \operatorname{Log}(n+x) - x \operatorname{Log} x - n + \tfrac{1}{2} \sum_{k=1}^{n} \operatorname{Log}^2(k+x) \right.$$

$$\left. - \tfrac{1}{2} n \operatorname{Log}^2(n+x) - \tfrac{1}{2}(\tfrac{1}{2}+x)\{\operatorname{Log}^2(n+x) - \operatorname{Log}^2 x\} \right).$$

Using this calculation in (18.15), we find that

$$\zeta''(0,x) = -2x + \operatorname{Log}^2 x + \lim_{n\to\infty} \left(-2n + 2(n+x) \operatorname{Log}(n+x) \right.$$

$$\left. -(n+x+\tfrac{1}{2}) \operatorname{Log}^2(n+x) + \sum_{k=1}^{n} \operatorname{Log}^2(k+x) \right).$$

In particular (see also (18.7)),

$$\zeta''(0,1) = -2 + \lim_{n\to\infty} \left(-2n + 2(n+1) \operatorname{Log}(n+1) \right.$$

$$\left. -(n+\tfrac{3}{2}) \operatorname{Log}^2(n+1) + \sum_{k=1}^{n} \operatorname{Log}^2(k+1) \right).$$

Hence, by (18.13), for $x > 0$,

$$\varphi(x-1) = 2x - \operatorname{Log}^2 x - 2 + \lim_{n\to\infty} \left((n+x+\tfrac{1}{2})\{\operatorname{Log}^2(n+x) \right.$$

$$- \operatorname{Log}^2(n+1)\} + (x-1) \operatorname{Log}^2(n+1) - 2(n+x+1) \operatorname{Log}\left(\frac{n+x}{n+1}\right)$$

$$+ 2 \operatorname{Log}(n+x) - 2x \operatorname{Log}(n+1)$$

$$\left. + \sum_{k=1}^{n} \{\operatorname{Log}^2(k+1) - \operatorname{Log}^2(k+x)\} \right)$$

$$= \lim_{n\to\infty} \left((x-1) \operatorname{Log}^2(n+1) + \sum_{k=0}^{n} \{\operatorname{Log}^2(k+1) - \operatorname{Log}^2(k+x)\} \right),$$

after a moderate calculation. Comparing this representation for $\varphi(x-1)$ with the far right side of (18.10), we see that the two definitions (18.10) and (18.13) are compatible.

Entry 18(ii). *Let* $\varphi(x)$ *be defined by* (18.13) *and let* C *be given by* (18.2). *If* n *is any natural number, then*

$$\varphi(x)-\sum_{j=0}^{n-1}\varphi\left(\frac{x-j}{n}\right)=2\operatorname{Log} n \operatorname{Log}\left(\frac{\Gamma(x+1)}{\sqrt{2\pi}}\right)-(x+\tfrac{1}{2})\operatorname{Log}^2 n-C(n-1). \tag{18.16}$$

Proof. Either (18.10) or (18.13) can be employed to prove (18.16), but the proof with (18.13) is computationally simpler.

Putting $r=(k+1)n-j$ below, we find that, for $\sigma>1$,

$$\begin{aligned}\zeta''(s,x+1)&-\sum_{j=0}^{n-1}\zeta''\left(s,\frac{x-j}{n}+1\right)\\&=\zeta''(s,x+1)-n^s\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\frac{\operatorname{Log}^2\{(nk+n-j+x)/n\}}{(nk+n-j+x)^s}\\&=\zeta''(s,x+1)-n^s\sum_{r=1}^{\infty}\frac{\operatorname{Log}^2\{(r+x)/n\}}{(r+x)^s}\\&=(1-n^s)\zeta''(s,x+1)-2n^s\operatorname{Log} n\,\zeta'(s,x+1)\\&\quad-n^s\operatorname{Log}^2 n\,\zeta(s,x+1).\end{aligned} \tag{18.17}$$

By analytic continuation, the extremal sides of (18.17) are equal in the entire complex s-plane. Using (18.13), (18.17) with $s=0$, and the fact that $\zeta''(0)=C$, we find that

$$\varphi(x)-\sum_{j=0}^{n-1}\varphi\left(\frac{x-j}{n}\right)=(1-n)C+2\operatorname{Log} n\,\zeta'(0,x+1)+\operatorname{Log}^2 n\,\zeta(0,x+1) \tag{18.18}$$

By (18.12) and the fact that $\zeta'(0)=-\frac{1}{2}\operatorname{Log}(2\pi)$ (Whittaker and Watson [1, p. 271]),

$$\zeta'(0,x+1)=\operatorname{Log}\left(\frac{\Gamma(x+1)}{\sqrt{2\pi}}\right).$$

Also, $\zeta(0,x+1)=-x-\frac{1}{2}$ (Whittaker and Watson [1, p. 271]). Using these evaluations in (18.18), we deduce (18.16).

Corollary. *Under the assumptions of Entry* 18 (ii), *we have*

$$\sum_{k=1}^{n-1}\varphi\left(-\frac{k}{n}\right)=\operatorname{Log}(2\pi)\operatorname{Log} n+\tfrac{1}{2}\operatorname{Log}^2 n+C(n-1).$$

Proof. Set $x=0$ in Entry 18(ii). Noting, by (18.13), that $\varphi(0)=0$, we deduce the desired result immediately.

Example 1. *We have*

$$\lim_{x\to\infty} x^{x\,\mathrm{Log}\,x-2x}e^{2x}\prod_{k=1}^{x}\frac{k^{1/k}}{k^{\mathrm{Log}\,k}}=(2\pi)^{\frac{1}{2}\mathrm{Log}(2\pi)}e^{-\frac{1}{2}\gamma^2+1/(24\pi^2)}.$$

Proof. Letting

$$P_x=\prod_{k=1}^{x}\frac{k^{1/k}}{k^{\mathrm{Log}\,k}},$$

we find that, by (17.5) and (18.3),

$$\begin{aligned}\mathrm{Log}\,P_x&=\sum_{k=1}^{x}\frac{\mathrm{Log}\,k}{k}-\sum_{k=1}^{x}\mathrm{Log}^2\,k\\&=-x\,\mathrm{Log}^2\,x+2x\,\mathrm{Log}\,x-2x-\tfrac{1}{2}\gamma^2+\tfrac{1}{24}\pi^2+\tfrac{1}{2}\,\mathrm{Log}^2(2\pi)+o(1),\end{aligned}$$

as x tends to ∞. After rearranging and exponentiating the last equality, we complete the proof.

Example 2. *Let* $\varphi(x)$ *be defined by* (18.13) *and let* C *be given by* (18.2). *Then*

$$\begin{aligned}\varphi(-\tfrac{1}{2})&=\mathrm{Log}(2\pi)\,\mathrm{Log}\,2+\tfrac{1}{2}\,\mathrm{Log}^2\,2+C,\\ \varphi(-\tfrac{1}{3})+\varphi(-\tfrac{2}{3})&=\mathrm{Log}(2\pi)\,\mathrm{Log}\,3+\tfrac{1}{2}\,\mathrm{Log}^2\,3+2C,\\ \varphi(-\tfrac{1}{4})+\varphi(-\tfrac{3}{4})&=\mathrm{Log}(2\pi)\,\mathrm{Log}\,2+\tfrac{3}{2}\,\mathrm{Log}^2\,2+2C,\end{aligned}$$

and

$$\varphi(-\tfrac{1}{6})+\varphi(-\tfrac{5}{6})=\mathrm{Log}\,2\,\mathrm{Log}\,3+2C.$$

Proof. The four evaluations above follow from putting $n=2$, 3, 4, and 6, respectively, in the last corollary.

In the second notebook, p. 98, Ramanujan writes "Find $\varphi(-\frac{1}{2})$, $\varphi(-\frac{1}{3})+\varphi(-\frac{2}{3})$, $\varphi(-\frac{1}{4})+\varphi(-\frac{3}{4})$, and $\varphi(-\frac{1}{6})+\varphi(-\frac{5}{6})$," but does not give their values. In the first notebook, p. 135, Ramanujan does record their values.

Entry 18(iii). *Let* $\varphi(x)$, c_1, *and* C *be defined by* (18.13), (17.2), *and* (18.2), *respectively. If* $0<x<1$, *then*

$$\begin{aligned}\tfrac{1}{2}\{\varphi(x-1)+\varphi(-x)\}&=c_1-\tfrac{1}{24}\pi^2\\&+\tfrac{1}{2}(\gamma+\mathrm{Log}(2\pi))(\gamma-\mathrm{Log}\{\tfrac{1}{2}\pi\csc^2(\pi x)\})-\sum_{k=1}^{\infty}\frac{\mathrm{Log}\,k\cos(2\pi kx)}{k}.\end{aligned}\tag{18.19}$$

Proof. From Hurwitz's formula (17.15), for $0<x<1$ and $\sigma<1$, we deduce that

$$\zeta(s,x)+\zeta(s,1-x)=4\Gamma(1-s)\sin(\tfrac{1}{2}\pi s)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{(2\pi k)^{1-s}}.$$

After two differentiations with respect to s, we find that

$$\begin{aligned}\zeta''(s, x) + \zeta''(s, 1-x) &= \{4\Gamma''(1-s)\sin(\tfrac{1}{2}\pi s) - 4\pi\Gamma'(1-s)\cos(\tfrac{1}{2}\pi s)\\ &\quad - \pi^2\Gamma(1-s)\sin(\tfrac{1}{2}\pi s)\}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{(2\pi k)^{1-s}} + \{-8\Gamma'(1-s)\sin(\tfrac{1}{2}\pi s)\\ &\quad + 4\pi\Gamma(1-s)\cos(\tfrac{1}{2}\pi s)\}\sum_{k=1}^{\infty}\frac{\operatorname{Log}(2\pi k)\cos(2\pi kx)}{(2\pi k)^{1-s}}\\ &\quad + 4\Gamma(1-s)\sin(\tfrac{1}{2}\pi s)\sum_{k=1}^{\infty}\frac{\operatorname{Log}^2(2\pi k)\cos(2\pi kx)}{(2\pi k)^{1-s}}.\end{aligned}$$

Letting $s = 0$ and using (18.13), we deduce that

$$\begin{aligned}\tfrac{1}{2}\{\varphi(x-1) + \varphi(-x)\} &= \Gamma'(1)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k} - \sum_{k=1}^{\infty}\frac{\operatorname{Log}(2\pi k)\cos(2\pi kx)}{k} + C\\ &= (\gamma + \operatorname{Log}(2\pi))\operatorname{Log}(2\sin(\pi x)) - \sum_{k=1}^{\infty}\frac{\operatorname{Log} k\cos(2\pi kx)}{k} + C,\end{aligned}$$

where we have used (17.13) and the fact that $\Gamma'(1) = -\gamma$. After some elementary manipulation, the formula above is transformed into (18.19).

For a further development of the theory of $\varphi(x)$ and for applications to the theory of real quadratic fields, see a paper of Deninger [1].

Before stating and proving the results in Section 19, we need to explain Ramanujan's terminology. For each nonnegative integer n, Ramanujan lets c_n denote the "constant" of $\sum_{k=1}^{x} \operatorname{Log}^n k$. In fact,

$$c_n = \lim_{x\to\infty}\left\{\sum_{k=1}^{x}\operatorname{Log}^n k - \int_1^x \operatorname{Log}^n t\,dt - \tfrac{1}{2}\operatorname{Log}^n x + (-1)^{n+1}n!\right\}. \quad (19.1)$$

By the Euler–Maclaurin formula (I3), this limit is easily seen to exist. Thus, $c_0 = -\frac{1}{2}$, $c_1 = \frac{1}{2}\operatorname{Log}(2\pi)$ (by 18.4), and $c_2 = C$ (by (18.2) and (18.5)). For each nonnegative integer n and positive integer x, define

$$\varphi_n(x) = \sum_{k=1}^{x}\operatorname{Log}^n k \quad \text{and} \quad \psi_n(x) = \varphi_n(x) - c_n. \quad (19.2)$$

Entry 19(i) concerns what Ramanujan calls "the logarithmic part" of $\psi_n(x)$. By the logarithmic part of $f(x)$, we shall mean that part of the asymptotic expansion of $f(x)$, as x tends to ∞, that involves positive powers of Log x multiplied by nonnegative powers of x. We denote the logarithmic part of $f(x)$ by $\mathscr{L}f(x)$.

Entry 19(i). *Let n be a positive integer. Then in the notation above,*

$$\mathscr{L}\psi_n(x) = \mathscr{L} \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \operatorname{Log}^k x\, \psi_{n-k}(x). \tag{19.3}$$

Proof. Using the Euler–Maclaurin formula (I5) and integrating by parts m times, we find that, as x tends to ∞,

$$\begin{aligned}\psi_m(x) &\sim \int_1^x \operatorname{Log}^m t\, dt + \tfrac{1}{2} \operatorname{Log}^m x \\ &\sim x \sum_{j=0}^{m} (-1)^j \frac{m!}{(m-j)!} \operatorname{Log}^{m-j} x + \tfrac{1}{2} \operatorname{Log}^m x,\end{aligned} \tag{19.4}$$

where m is any nonnegative integer. In particular,

$$\mathscr{L}\psi_n(x) = (x+\tfrac{1}{2}) \operatorname{Log}^n x + x \sum_{j=1}^{n-1} (-1)^j \frac{n!}{(n-j)!} \operatorname{Log}^{n-j} x. \tag{19.5}$$

Using (19.4), we find that

$$\begin{aligned}&\mathscr{L} \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \operatorname{Log}^k x\, \psi_{n-k}(x) \\ &= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \operatorname{Log}^k x \Bigg\{(x+\tfrac{1}{2}) \operatorname{Log}^{n-k} x \\ &\quad + x \sum_{j=1}^{n-k} (-1)^j \frac{(n-k)!}{(n-k-j)!} \operatorname{Log}^{n-k-j} x\Bigg\} \\ &= (x+\tfrac{1}{2}) \operatorname{Log}^n x \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} + x \sum_{j=1}^{n-1} (-1)^j \operatorname{Log}^{n-j} x \sum_{k=1}^{n-j} \frac{(-1)^{k+1} n!}{k!\,(n-k-j)!} \\ &= (x+\tfrac{1}{2}) \operatorname{Log}^n x + x \sum_{j=1}^{n-1} (-1)^j \frac{n!}{(n-j)!} \operatorname{Log}^{n-j} x \sum_{k=1}^{n-j} (-1)^{k+1} \binom{n-j}{k} \\ &= (x+\tfrac{1}{2}) \operatorname{Log}^n x + x \sum_{j=1}^{n-1} (-1)^j \frac{n!}{(n-j)!} \operatorname{Log}^{n-j} x.\end{aligned} \tag{19.6}$$

Comparing (19.5) and (19.6), we deduce (19.3).

Entry 19(ii). *Let $\psi_n(x)$, $n \geq 0$, be defined by* (19.2). *Then as x tends to ∞,*

$$\begin{aligned}&\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \operatorname{Log}^k x\, \psi_{n-k}(x) \sim n!\, x \\ &\quad - \frac{B_{n+1}}{(n+1)x^n} - \frac{nB_{n+2}}{2(n+2)x^{n+1}} - \frac{n(n+\frac{5}{3})B_{n+3}}{2^2 2!\,(n+3)x^{n+2}} \\ &\quad - \frac{n(n+2)(n+3)B_{n+4}}{2^3 3!\,(n+4)x^{n+3}} - \frac{n\{(n+2)(n+4)^2 + \frac{1}{3}(n+2) + \frac{4}{5}\} B_{n+5}}{2^4 4!\,(n+5)x^{n+4}} \\ &\quad - \frac{n(n+4)(n+5)\{(n+3)(n+4) + \frac{2}{3}(n+1)\} B_{n+6}}{2^5 5!\,(n+6)x^{n+5}} + \cdots.\end{aligned} \tag{19.7}$$

Proof. If $n=0$, the theorem is verified at once. Thus, in the sequel, we assume that $n \geq 1$. From the Euler–Maclaurin formula (I5), we find that, as x tends to ∞,

$$\begin{aligned}
&\sum_{k=0}^{n} (-1)^{n+k}\binom{n}{k} \operatorname{Log}^k x\, \psi_{n-k}(x) \\
&\sim \sum_{k=0}^{n} (-1)^{n+k}\binom{n}{k} \operatorname{Log}^k x \left\{ (x+\tfrac{1}{2}) \operatorname{Log}^{n-k} x \right. \\
&\quad \left. + x\sum_{j=1}^{n-k} (-1)^j \frac{(n-k)!}{(n-k-j)!} \operatorname{Log}^{n-k-j} x + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} \frac{d^{2j-1} \operatorname{Log}^{n-k} x}{dx^{2j-1}} \right\} \\
&= x\sum_{j=1}^{n} (-1)^{n+j} \frac{n!}{(n-j)!} \operatorname{Log}^{n-j} x \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} + R_n \\
&= xn! + R_n,
\end{aligned}$$

where

$$R_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \operatorname{Log}^{n-k} x \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} \frac{d^{2j-1} \operatorname{Log}^k x}{dx^{2j-1}}. \tag{19.8}$$

It remains to calculate the terms in the asymptotic expansion R_n that are indicated by (19.7). We shall repeatedly employ the evaluations

$$\sum_{k=0}^{m} (-1)^k k^r \binom{m}{k} = \begin{cases} 0, & 0 \leq r < m, \\ (-1)^m m!, & r = m. \end{cases} \tag{19.9}$$

First, from (19.9), we observe that if $2j-1 < n$, then the contribution to R_n of the jth term of the inner sum in (19.8) is equal to 0.

Suppose that n is odd. Appealing to (19.9), we find that when $2j-1=n$, the contribution from (19.8) is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^n}{x^n} \frac{B_{n+1}}{(n+1)!} = -\frac{n!\, B_{n+1}}{x^n (n+1)!} = -\frac{B_{n+1}}{x^n(n+1)}. \tag{19.10}$$

If n is even, there is no value of j for which $2j-1=n$. But then $B_{n+1}=0$. Thus, in either case, (19.10) is in agreement with (19.7).

If $n+1$ is odd, the next contribution from (19.8) will be

$$-\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{n(n+1)k^n B_{n+2}}{2x^{n+1}(n+2)!} = -\frac{nB_{n+2}}{2(n+2)x^{n+1}},$$

by (19.9). Thus, whether n is even or odd, we see that this second contribution above is in compliance with (19.7).

Whether n is even or odd, the third contribution in (19.8) is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^n B_{n+3}}{x^{n+2}(n+3)!} \sum_{\substack{i,j=1 \\ i<j}}^{n+1} ij. \tag{19.11}$$

Now (Abramowitz and Stegun [1, p. 1])

$$\sum_{\substack{i,j=1\\ i<j}}^{n+1} ij = \sum_{j=1}^{n+1} \frac{(j-1)j^2}{2}$$

$$= \frac{1}{2}\left\{\frac{(n+1)^2(n+2)^2}{4} - \frac{(n+1)(n+2)(2n+3)}{6}\right\}$$

$$= \tfrac{1}{8}n(n+1)(n+2)(n+\tfrac{5}{3}).$$

Thus, (19.11) becomes, with the help of (19.9),

$$-\frac{n!\,B_{n+3}n(n+1)(n+2)(n+\frac{5}{3})}{8(n+3)!\,x^{n+2}} = -\frac{n(n+\frac{5}{3})B_{n+3}}{8(n+3)x^{n+2}},$$

which is compatible with the asymptotic expansion (19.7).

The remaining calculations are facilitated by employing a theorem of Faà di Bruno [1, p. 12]. Let

$$S_r(n) = \sum_{j=1}^{n} j^r$$

and

$$A_k(n) = \sum_{\substack{i_2, i_2, \ldots, i_k = 1\\ i_1 < i_2 < \cdots < i_k}}^{n} i_1 i_2 \ldots i_k.$$

Then for $k \geq 1$,

$$A_k(n) = \frac{1}{k!}\begin{vmatrix} S_1(n) & 1 & 0 & 0 & \cdots & 0 \\ S_2(n) & S_1(n) & 2 & 0 & \cdots & 0 \\ S_3(n) & S_2(n) & S_1(n) & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ S_k(n) & S_{k-1}(n) & S_{k-2}(n) & S_{k-3}(n) & \cdots & S_1(n) \end{vmatrix}. \tag{19.12}$$

The values for $S_j(n)$, $1 \leq j \leq 5$, that we shall need below may be found in Abramowitz and Stegun [1, pp. 1, 2].

Returning to (19.8), we see that the fourth contribution is

$$-\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^n B_{n+4} A_3(n+2)}{(n+4)!\,x^{n+3}}. \tag{19.13}$$

By (19.12),

$$A_3(n+2) = \tfrac{1}{6}S_1^3(n+2) - \tfrac{1}{2}S_1(n+2)S_2(n+2) + \tfrac{1}{3}S_3(n+2)$$

$$= \frac{(n+2)^3(n+3)^3}{48} - \frac{(n+2)^2(n+3)^2(2n+5)}{24} + \frac{(n+2)^2(n+3)^2}{12}$$

$$= \frac{n(n+1)(n+2)^2(n+3)^2}{48}.$$

Thus, (19.13) becomes, with the aid of (19.9),

$$-\frac{n!\ B_{n+4}n(n+1)(n+2)^2(n+3)^2}{48(n+4)!\ x^{n+3}} = -\frac{n(n+2)(n+3)B_{n+4}}{48(n+4)x^{n+3}},$$

which coincides with the appropriate term in (19.7).

The fifth contribution from (19.8) is

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^nB_{n+5}A_4(n+3)}{(n+5)!\ x^{n+4}}. \tag{19.14}$$

By (19.12),

$$\begin{aligned}A_4(n+3) &= \tfrac{1}{24}S_1^4(n+3) - \tfrac{1}{4}S_1^2(n+3)S_2(n+3) + \tfrac{1}{8}S_2^2(n+3)\\ &\quad + \tfrac{1}{3}S_1(n+3)S_3(n+3) - \tfrac{1}{4}S_4(n+3)\\ &= \frac{(n+3)^4(n+4)^4}{2^73} - \frac{(n+3)^3(n+4)^3(2n+7)}{2^53}\\ &\quad + \frac{(n+3)^2(n+4)^2(2n+7)^2}{2^53^2} + \frac{(n+3)^3(n+4)^3}{2^33}\\ &\quad - \frac{(n+3)(n+4)(2n+7)\{3(n+3)^2+3(n+3)-1\}}{2^33\cdot 5}\\ &= \frac{n(n+1)(n+2)(n+3)(n+4)}{2^44!}\left(n^3+10n^2+\frac{97}{3}n+\frac{502}{15}\right),\end{aligned}$$

after an extremely laborious calculation. Thus, (19.14) becomes, with the help of (19.9),

$$\begin{aligned}&-\frac{n!\ B_{n+5}n(n+1)(n+2)(n+3)(n+4)(n^3+10n^2+\frac{97}{3}n+\frac{502}{15})}{(n+5)!\ 2^44!\ x^{n+4}}\\ &\qquad\qquad = -\frac{nB_{n+5}\{(n+2)(n+4)^2+\frac{1}{3}(n+2)+\frac{4}{5}\}}{2^44!\ (n+5)x^{n+4}},\end{aligned}$$

which coincides with the penultimate displayed term on the right side of (19.7).

Lastly, the sixth contribution from (19.8) is

$$-\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^nB_{n+6}A_5(n+4)}{(n+6)!\ x^{n+5}}. \tag{19.15}$$

Suppressing the argument $n+4$, we find that (19.12) yields

$$\begin{aligned}A_5 &= \tfrac{1}{120}S_1^5 - \tfrac{1}{12}S_1^3S_2 + \tfrac{1}{8}S_1S_2^2 + \tfrac{1}{6}S_1^2S_3 - \tfrac{1}{6}S_2S_3 - \tfrac{1}{4}S_1S_4 + \tfrac{1}{5}S_5\\ &= \frac{(n+4)^5(n+5)^5}{120\cdot 32} - \frac{(n+4)^4(n+5)^4(2n+9)}{12\cdot 8\cdot 6}\\ &\quad + \frac{(n+4)^3(n+5)^3(2n+9)^2}{8\cdot 2\cdot 36} + \frac{(n+4)^4(n+5)^4}{6\cdot 4\cdot 4}\end{aligned}$$

$$-\frac{(n+4)^3(n+5)^3(2n+9)}{6\cdot 6\cdot 4}$$

$$-\frac{(n+4)^2(n+5)^2(2n+9)\{3(n+4)^2+3(n+4)-1\}}{4\cdot 2\cdot 30}$$

$$+\frac{(n+4)^2(n+5)^2\{2(n+4)^2+2(n+4)-1\}}{5\cdot 12}$$

$$=\frac{(n+4)^2(n+5)^2n(n+1)(n+2)(n+3)(n^2+\frac{23}{3}n+\frac{38}{3})}{2^5 5!},$$

after an excruciatingly tedious computation. Thus, the sixth contribution (19.15) is, by (19.9),

$$-\frac{n!\ B_{n+6}(n+4)^2(n+5)^2n(n+1)(n+2)(n+3)\{(n+3)(n+4)+\frac{2}{3}(n+1)\}}{2^5 5!\ (n+6)!\ x^{n+5}},$$

which easily reduces to the last displayed term in (19.7).

In analogy with the previous two sections, Ramanujan now examines analytic extensions of $\varphi_n(x)$ and $\psi_n(x)$ but does not indicate their definitions. In view of (18.12) and (18.13), we shall define, for every real value of x and positive integer n,

$$\varphi_n(x)=(-1)^{n+1}\{\zeta^{(n)}(0,\ x+1)-\zeta^{(n)}(0)\}. \tag{19.16}$$

Thus, we want to let

$$\psi_n(x)=(-1)^{n+1}\zeta^{(n)}(0,\ x+1),\qquad n\geq 1. \tag{19.17}$$

We first show that (19.16) and (19.17) are consistent with the previous definitions (19.2) made for just positive, integral values of x. For $\sigma>1$, a simple calculation shows that

$$(-1)^{n+1}\{\zeta^{(n)}(s,\ x+1)-\zeta^{(n)}(s,\ x)\}=x^{-s}\operatorname{Log}^n x,$$

which, by analytic continuation, is valid for all complex s. Thus, by (19.16),

$$\varphi_n(x)-\varphi_n(x-1)=\operatorname{Log}^n x,$$

and since $\varphi_n(0)=0$, $\varphi_n(x)=\sum_{k=1}^{x}\operatorname{Log}^n k$, for each positive integer x, as desired.

Secondly, we need to show that $c_n=(-1)^n\zeta^{(n)}(0)$, where c_n is defined by (19.1). Using (18.6), it is easy to show by induction that

$$\zeta^{(n)}(s)=(-1)^{n-1}n\int_1^\infty\frac{[x]-x+\frac{1}{2}}{x^{s+1}}\operatorname{Log}^{n-1}x\,dx$$

$$+(-1)^n s\int_1^\infty\frac{[x]-x+\frac{1}{2}}{x^{s+1}}\operatorname{Log}^n x\,dx+\frac{(-1)^n n!}{(s-1)^{n+1}}, \tag{19.18}$$

where $n \geq 1$ and $\sigma > -1$. Thus,

$$\zeta^{(n)}(0) = (-1)^{n-1} n \int_1^\infty \frac{[x] - x + \frac{1}{2}}{x} \operatorname{Log}^{n-1} x\, dx - n!$$

$$= (-1)^{n-1} n \lim_{N\to\infty} \left(\sum_{k=1}^{N-1} k \int_k^{k+1} \frac{\operatorname{Log}^{n-1} x}{x} dx - \int_1^N \operatorname{Log}^{n-1} x\, dx \right.$$

$$\left. + \frac{1}{2} \int_1^N \frac{\operatorname{Log}^{n-1} x}{x} dx \right) - n!$$

$$= (-1)^{n-1} \lim_{N\to\infty} \left(\sum_{k=1}^{N-1} k\{\operatorname{Log}^n(k+1) - \operatorname{Log}^n k\} \right.$$

$$\left. - N \operatorname{Log}^n N + \int_1^N \operatorname{Log}^n x\, dx + \tfrac{1}{2} \operatorname{Log}^n N \right) - n!.$$

Thus,

$$(-1)^n \zeta^{(n)}(0) = \lim_{N\to\infty} \left(\sum_{k=1}^{N} \operatorname{Log}^n k - \int_1^N \operatorname{Log}^n x\, dx - \tfrac{1}{2} \operatorname{Log}^n N \right) - (-1)^n\, n!.$$

Hence, by (19.1), $(-1)^n \zeta^{(n)}(0) = c_n$, $n \geq 1$, as we wished to show.

Entry 19(iii). *Let $\psi_n(x)$ be defined by (19.17), where $n \geq 1$. If r is any positive integer, then*

$$\psi_n(x) - \sum_{j=0}^{r-1} \psi_n\left(\frac{x-j}{r}\right) = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} \operatorname{Log}^j r\, \psi_{n-j}(x). \quad (19.19)$$

Before proving Entry 19(iii), we note two special cases. If $n = 1$, then $\psi_1(x) = \operatorname{Log} \Gamma(x+1) - \frac{1}{2} \operatorname{Log}(2\pi)$. Since $\psi_0(x) = x + \frac{1}{2}$, Entry 19(iii) yields

$$\operatorname{Log} \Gamma(x+1) - \sum_{j=0}^{r-1} \operatorname{Log} \Gamma\left(\frac{x-j}{r} + 1\right) = (x + \tfrac{1}{2}) \operatorname{Log} r - \tfrac{1}{2}(r-1) \operatorname{Log}(2\pi),$$

which is just another form of Gauss's multiplication theorem (Whittaker and Watson [1, p. 240]). The case $n = 2$ of (19.19) is equivalent to Entry 18(ii).

Proof. Let $\sigma > 1$. Setting $m = kr + r - j$ below, we find that

$$(-1)^{n+1} \zeta^{(n)}(s, x+1) - \sum_{j=0}^{r-1} (-1)^{n+1} \zeta^{(n)}\left(s, \frac{x-j}{r} + 1\right)$$

$$= (-1)^{n+1} \zeta^{(n)}(s, x+1) + r^s \sum_{j=0}^{r-1} \sum_{k=0}^{\infty} \frac{\operatorname{Log}^n\{(x-j+r+kr)/r\}}{(x-j+r+kr)^s}$$

$$= (-1)^{n+1} \zeta^{(n)}(s, x+1) + r^s \sum_{m=1}^{\infty} \frac{\operatorname{Log}^n\{(x+m)/r\}}{(x+m)^s}$$

$$= (-1)^{n+1} \zeta^{(n)}(s, x+1)(1 - r^s)$$

$$+ \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} \operatorname{Log}^j r\, (-1)^{n-j+1} \zeta^{(n-j)}(s, x+1). \quad (19.20)$$

By analytic continuation, the extremal sides of (19.20) are equal for all complex numbers s. If we set $s=0$ in (19.20) and use (19.17), then we obtain precisely the equality (19.19).

Corollary 1. *For each pair of positive integers n, r, we have*

$$\sum_{j=1}^{r-1} \psi_n\left(-\frac{j}{r}\right) = \sum_{j=1}^{n} \binom{n}{j}(-1)^{j+1} c_{n-j} \operatorname{Log}^j r.$$

Proof. Put $x=0$ in (19.19). Since $\psi_k(0) = -c_k$, $k \geq 0$, the desired result follows.

Ramanujan's version of Corollary 1 contains an extra additive factor of $-c_n$ on the right side, p. 99.

Corollary 2. *Let $\mathscr{L}f(x)$ denote the logarithmic part of $f(x)$ as defined prior to Entry* 19(i). *If x and n are positive integers, then*

$$\mathscr{L} \sum_{j=0}^{x-1} \psi_n\left(\frac{x-j}{x}\right) = 0.$$

Proof. Put $r=x$ in Entry 19(iii) to get

$$\sum_{j=0}^{x-1} \psi_n\left(\frac{x-j}{x}\right) = \sum_{j=0}^{n} \binom{n}{j}(-1)^j \operatorname{Log}^j x\, \psi_{n-j}(x).$$

Thus, by (19.3),

$$\mathscr{L} \sum_{j=0}^{x-1} \psi_n\left(\frac{x-j}{x}\right) = \mathscr{L} \sum_{j=0}^{n} \binom{n}{j}(-1)^j \operatorname{Log}^j x\, \psi_{n-j}(x) = 0.$$

Entry 20(i). *Let r and x denote positive integers and suppose that $k > 0$. Define $F_k(x)$ and $\varphi(x, k) = \varphi(x)$ by*

$$F_k(x) = k^x \varphi(x) = \sum_{n=1}^{x} n^r k^{n-1}. \tag{20.1}$$

If $k \neq 1$, then

$$\varphi(x) = \sum_{j=0}^{r} \binom{r}{j} \frac{(-1)^j \psi_j(-k) x^{r-j}}{(k-1)^{j+1}} + \frac{C_r(k)}{k^x} + \frac{x^r}{2k} + \frac{R}{k^{x+1}}, \tag{20.2}$$

where

$$\psi_j(-k) = \frac{j!\,(k-1)^{j+1}}{k \operatorname{Log}^{j+1} k}, \qquad 0 \leq j \leq r, \tag{20.3}$$

$$C_r(k) = \frac{(-1)^{r+1}\, r!}{k \operatorname{Log}^{r+1} k}, \tag{20.4}$$

and

$$R = \int_0^x P_1(t)k^t(rt^{r-1} + t^r \operatorname{Log} k)\, dt. \tag{20.5}$$

Entry 20(i) is not given very explicitly by Ramanujan. The last two expressions on the right side of (20.2) are missing in the notebooks, p. 99. The definition of $C_r(k)$ is not given and can only be inferred from the next entry. Moreover, Ramanujan has written $C_r(k)$ instead of $C_r(k)/k^x$. Lastly, Ramanujan writes "where ψ is the same ψ in." The sentence is not completed, and so the definition (20.3) of $\psi_j(-k)$ is not given.

Proof. Applying the Euler–Maclaurin summation formula (13), we find that

$$k^{x+1}\varphi(x) = \sum_{n=1}^{x} n^r k^n = \int_0^x t^r k^t\, dt + \tfrac{1}{2}x^r k^x + R, \tag{20.6}$$

where R is given by (20.5). Upon successively integrating by parts, we get

$$\int_0^x t^r k^t\, dt = \sum_{j=0}^{r} \frac{(-1)^j r!\, k^x x^{r-j}}{(r-j)!\, \operatorname{Log}^{j+1} k} - \frac{(-1)^r r!}{\operatorname{Log}^{r+1} k}. \tag{20.7}$$

Substituting (20.7) into (20.6) and employing the notation (20.3) and (20.4), we readily deduce (20.2).

Entry 20(ii). *For each positive integer* r,

$$C_r(k) = \frac{\psi_r(-k)}{(1-k)^{r+1}} \quad \text{and} \quad k\psi_j(-k) = k^j \psi_j\left(-\frac{1}{k}\right), \qquad 0 \le j \le r.$$

The latter equality is stated by Ramanujan for only $j = r$, and the subscript on the right side is inadvertently omitted by him.

Proof. The first equality is immediately verified by using (20.4) and (20.3) with $j = r$.

The second equality is an easy consequence of (20.3).

Ramanujan next examines an extension of $F_k(x)$ for all real values of x, but he does not divulge his definition. Following previous practices of Ramanujan (see (17.7), for example), assume that $0 < k < 1$ and define, for x and r real,

$$F_k(x) = \sum_{n=1}^{\infty} \{n^r k^{n-1} - (n+x)^r k^{n+x-1}\}. \tag{20.8}$$

An elementary calculation shows that

$$F_k(x) - F_k(x-1) = x^r k^{x-1},$$

and so F_k satisfies the natural difference equation. In particular, if x is a positive integer, we see that (20.1) is valid.

Although he uses the same notation, the definition of the constant $C_r(k)$ in

Entry 20(iii) is not the same as in Entries 20(i), (ii). In Chapter 6, we pointed out that Ramanujan's definition of the "constant" of a series is deficient if the series actually converges. We seem to have an example of this here.

Entry 20(iii). *For* $0<k<1$ *and r real, let*

$$C_r(k)=\sum_{m=1}^{\infty} m^r k^{m-1}.$$

If x is real and n is any positive integer, then

$$\sum_{j=0}^{n-1} F_k\left(\frac{x-j}{n}\right)=nC_r(k)+n^{-r}k^{(1-n)/n}\{F_{k^{1/n}}(x)-C_r(k^{1/n})\}, \tag{20.9}$$

where F_k *is defined by* (20.8).

Proof. Replacing $mn-j$ by m below, we find that

$$\begin{aligned}\sum_{j=0}^{n-1} F_k\left(\frac{x-j}{n}\right) &= \sum_{j=0}^{n-1}\sum_{m=1}^{\infty}\left\{m^r k^{m-1}-\left(m+\frac{x-j}{n}\right)^r k^{m+(x-j)/n-1}\right\}\\ &= nC_r(k)-n^{-r}\sum_{m=1}^{\infty}(m+x)^r k^{(m+x)/n-1}\\ &= nC_r(k)+n^{-r}k^{(1-n)/n}\left(\sum_{m=1}^{\infty}\{m^r k^{(m-1)/n}\right.\\ &\qquad \left.-(m+x)^r k^{(m+x-1)/n}\}-\sum_{m=1}^{\infty} m^r k^{(m-1)/n}\right),\end{aligned}$$

which yields (20.9) immediately.

Corollary. *Under the same hypotheses as Entry* 20(iii), *we have*

$$\sum_{j=1}^{n-1} F_k\left(-\frac{j}{n}\right)=nC_r(k)-n^{-r}k^{(1-n)/n}C_r(k^{1/n}).$$

Proof. Set $x=0$ in (20.9).

In Entry 21(i) Ramanujan considers

$$\varphi_r(x)\equiv\sum_{k=1}^{x}\frac{\operatorname{Log} k}{k^r}, \tag{21.1}$$

where apparently he assumes that $r>1$. The case $r=1$ was considered in Section 17, and the case $r<0$ is examined in Chapter 9.

Entry 21(i). *Define*

$$H_n(r)=\sum_{k=0}^{n}\frac{1}{k+r}. \tag{21.2}$$

Let $\zeta(s, x)$ denote the extended version of the Hurwitz zeta-function defined by (18.11). *Recall that $(r)_n$ is defined in* (I7). *Then as x tends to ∞,*

$$\varphi_r(x) + \operatorname{Log} x\, \zeta(r, x+1) \sim -\zeta'(r) - \frac{1}{(r-1)^2 x^{r-1}} + \sum_{k=1}^{\infty} \frac{B_{2k}(r)_{2k-1} H_{2k-2}(r)}{(2k)!\, x^{r+2k-1}}. \tag{21.3}$$

Proof. Applying the Euler–Maclaurin summation formula (I5), we find that, as x tends to ∞,

$$\varphi_r(x) \sim \int_1^x t^{-r} \operatorname{Log} t\, dt + \frac{\operatorname{Log} x}{2x^r} + c' + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x), \tag{21.4}$$

where $f(x) = x^{-r} \operatorname{Log} x$ and c' is some constant.

Firstly, integrating by parts, we find that

$$\int_1^x t^{-r} \operatorname{Log} t\, dt = \frac{\operatorname{Log} x}{(1-r)x^{r-1}} - \frac{1}{(1-r)^2 x^{r-1}} + \frac{1}{(1-r)^2}. \tag{21.5}$$

Secondly, put $f_1(x) = \operatorname{Log} x$ and $f_2(x) = x^{-r}$. Then

$$f_1^{(k)}(x) = (-1)^{k+1}(k-1)!\, x^{-k}, \qquad k \geq 1,$$

and

$$f_2^{(k)}(x) = (-1)^k (r)_k x^{-r-k}, \qquad k \geq 0.$$

Applying Leibniz's rule, we find that, for each positive integer n,

$$f^{(n)}(x) = \frac{(-1)^n (r)_n \operatorname{Log} x}{x^{r+n}} + \frac{(-1)^{n+1}\, n!}{x^{r+n}} \sum_{k=1}^{n} \frac{(r)_{n-k}}{(n-k)!\, k}.$$

For arbitrary a and each positive integer n (see, e.g., Hansen's tables [1, p. 126]),

$$\sum_{k=0}^{n-1} \frac{(a)_k}{k!\,(n-k)} = \frac{(a)_n}{n!} H_{n-1}(a), \tag{21.6}$$

where $H_n(r)$ is defined by (21.2). Thus,

$$f^{(n)}(x) = \frac{(-1)^n (r)_n \operatorname{Log} x}{x^{r+n}} + \frac{(-1)^{n+1}(r)_n H_{n-1}(r)}{x^{r+n}}. \tag{21.7}$$

Employing (21.5) and (21.7) in (21.4), we deduce that

$$\varphi_r(x) \sim c + \frac{\operatorname{Log} x}{2x^r} + \frac{\operatorname{Log} x}{(1-r)x^{r-1}} - \frac{1}{(r-1)^2 x^{r-1}} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left\{ -\frac{(r)_{2k-1} \operatorname{Log} x}{x^{r+2k-1}} + \frac{(r)_{2k-1} H_{2k-2}(r)}{x^{r+2k-1}} \right\}, \tag{21.8}$$

as x tends to ∞, where c is some constant. Since $r > 1$, we see, by letting x tend to ∞ in (21.8), that $c = -\zeta'(r)$. Now by Entry 26 of Chapter 5,

$$\zeta(r, x+1) \sim \frac{1}{(r-1)x^{r-1}} - \frac{1}{2x^r} + \sum_{k=1}^{\infty} \frac{B_{2k}(r)_{2k-1}}{(2k)!\, x^{r+2k-1}}, \tag{21.9}$$

as x tends to ∞. Employing (21.9) in (21.8) and using the fact $c = -\zeta'(r)$, we deduce (21.3) to complete the proof.

We now extend the definition (21.1) of $\varphi_r(x)$ to all real values of x and positive values of r by defining

$$\varphi_r(x) = \sum_{k=1}^{\infty} \left\{ \frac{\operatorname{Log} k}{k^r} - \frac{\operatorname{Log}(k+x)}{(k+x)^r} \right\}. \tag{21.10}$$

An elementary calculation shows that (21.1) and (21.10) are compatible.

Entry 21(ii). *Let $\varphi_r(x)$ be defined by* (21.10). *Then for $r > 0$ and $|x| < 1$,*

$$\varphi_r(x) = \sum_{j=1}^{\infty} \binom{-r}{j} \{\zeta(r+j)H_{j-1}(r) - \zeta'(r+j)\} x^j,$$

where $H_n(r)$ is defined by (21.2).

Proof. By (21.10),

$$\varphi_r(x) = \sum_{k=1}^{\infty} \operatorname{Log} k \left\{ \frac{1}{k^r} - \frac{1}{(k+x)^r} \right\} - \sum_{k=1}^{\infty} \frac{\operatorname{Log}(1+x/k)}{(k+x)^r}$$
$$= S_1 + S_2, \tag{21.11}$$

say,

First, for $|x| < 1$,

$$S_1 = \sum_{k=1}^{\infty} \operatorname{Log} k \left\{ \frac{1}{k^r} - \frac{1}{k^r} \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{x}{k}\right)^j \right\}$$
$$= -\sum_{k=1}^{\infty} \frac{\operatorname{Log} k}{k^r} \sum_{j=1}^{\infty} \binom{-r}{j} \left(\frac{x}{k}\right)^j$$
$$= \sum_{j=1}^{\infty} \binom{-r}{j} \zeta'(r+j) x^j, \tag{21.12}$$

where we have inverted the order of summation by absolute convergence.

Secondly, for $|x| < 1$,

$$S_2 = -\sum_{k=1}^{\infty} \frac{1}{k^r} \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{x}{k}\right)^j \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\frac{x}{k}\right)^m$$
$$= \sum_{k=1}^{\infty} \frac{1}{k^r} \sum_{n=1}^{\infty} (-1)^n \sum_{j=0}^{n-1} \frac{(r)_j}{j!(n-j)} \left(\frac{x}{k}\right)^n$$
$$= \sum_{k=1}^{\infty} \frac{1}{k^r} \sum_{n=1}^{\infty} \frac{(-1)^n (r)_n H_{n-1}(r)}{n!} \left(\frac{x}{k}\right)^n$$
$$= \sum_{n=1}^{\infty} \binom{-r}{n} \zeta(r+n) H_{n-1}(r) x^n, \tag{21.13}$$

where in the penultimate equality we applied (21.6), and where again we inverted the order of summation by absolute convergence. Putting (21.12) and (21.13) in (21.11), we finish the proof.

Entry 21(iii). *If $r>1$ and n is any natural number, then*

$$n^r\varphi_r(x)-\sum_{j=0}^{n-1}\varphi_r\left(\frac{x-j}{n}\right)=(n-n^r)\zeta'(r)-n^r \operatorname{Log} n\, \zeta(r, x+1).$$

Proof. By (21.10),

$$\begin{aligned}
n^r\varphi_r(x)&-\sum_{j=0}^{n-1}\varphi_r\left(\frac{x-j}{n}\right)\\
&=\sum_{k=1}^{\infty}\left\{n^r\left(\frac{\operatorname{Log} k}{k^r}-\frac{\operatorname{Log}(k+x)}{(k+x)^r}\right)-\sum_{j=0}^{n-1}\left(\frac{\operatorname{Log} k}{k^r}-\frac{\operatorname{Log}\{(kn-j+x)/n\}}{\{(kn-j+x)/n\}^r}\right)\right\}\\
&=(n-n^r)\zeta'(r)-\sum_{k=1}^{\infty}\left\{n^r\frac{\operatorname{Log}(k+x)}{(k+x)^r}-\frac{\operatorname{Log}\{(k+x)/n\}}{\{(k+x)/n\}^r}\right\}\\
&=(n-n^r)\zeta'(r)-n^r \operatorname{Log} n\sum_{k=1}^{\infty}\frac{1}{(k+x)^r},
\end{aligned} \tag{21.14}$$

and the proof is complete.

Corollary. *For $r>1$ and each positive integer n,*

$$\sum_{j=1}^{n-1}\varphi_r\left(-\frac{j}{n}\right)=n^r \operatorname{Log} n\,\zeta(r)+(n^r-n)\zeta'(r).$$

Proof. Set $x=0$ in Entry 21(iii).

Ramanujan begins Section 22 by defining

$$\varphi_r(x)=\sum_{k=1}^{x}\frac{\operatorname{Log}^r k}{k}, \qquad r\geq 0. \tag{22.1}$$

Entry 22(i). *If c_r is the constant appearing in the asymptotic expansion of $\varphi_r(x)$ as x tends to ∞, then*

$$c_r=\lim_{x\to\infty}\left\{\varphi_r(x)-\frac{\operatorname{Log}^{r+1} x}{r+1}\right\}.$$

Proof. From the Euler–Maclaurin summation formula (I3), as x tends to ∞,

$$\varphi_r(x)=\int_1^x\frac{\operatorname{Log}^r t}{t}\,dt+c_r+o(1),$$

and the desired result follows forthwith.

We now extend the definition of $\varphi_r(x)$ by defining

$$\varphi_r(x)=\sum_{k=1}^{\infty}\left\{\frac{\operatorname{Log}^r k}{k}-\frac{\operatorname{Log}^r(k+x)}{k+x}\right\} \tag{22.2}$$

for all real values of x and $r \geq 0$. It is easy to see that (22.1) and (22.2) are compatible. Note that the case $r = 1$ was studied in Section 17. The next result generalizes Entry 17(iii).

Entry 22(ii). *Let r be any nonnegative integer and n any positive integer. With* $\varphi_r(x)$ defined by (22.2), *then*

$$n\varphi_r(x) - \sum_{j=0}^{n-1} \varphi_r\left(\frac{x-j}{n}\right)$$

$$= \frac{(-1)^r n}{r+1} \operatorname{Log}^{r+1} n + n \sum_{j=1}^{r} \binom{r}{j} (-1)^{j+1} \operatorname{Log}^j n\{\varphi_{r-j}(x) - c_{r-j}\},$$

where c_k is given in Entry 22(i).

Proof. For any positive integer m, consider

$$n\sum_{k=1}^{mn}\left\{\frac{\operatorname{Log}^r k}{k} - \frac{\operatorname{Log}^r(k+x)}{k+x}\right\} - \sum_{j=0}^{n-1}\sum_{k=1}^{m}\left\{\frac{\operatorname{Log}^r k}{k} - \frac{\operatorname{Log}^r(k+(x-j)/n)}{k+(x-j)/n}\right\}$$

$$= n\sum_{k=1}^{mn}\left\{\frac{\operatorname{Log}^r k}{k} - \frac{\operatorname{Log}^r(k+x)}{k+x}\right\} - n\sum_{k=1}^{m}\frac{\operatorname{Log}^r k}{k} + n\sum_{k=1}^{mn}\frac{\operatorname{Log}^r((k+x)/n)}{k+x}$$

$$= n\left(\sum_{k=1}^{mn}\frac{\operatorname{Log}^r k}{k} - \sum_{k=1}^{m}\frac{\operatorname{Log}^r k}{k}\right) + n\sum_{k=1}^{mn}\sum_{j=1}^{r}\binom{r}{j}\frac{(-1)^j \operatorname{Log}^j n \operatorname{Log}^{r-j}(k+x)}{k+x}$$

$$= n\left(\sum_{k=1}^{mn}\frac{\operatorname{Log}^r k}{k} - \sum_{k=1}^{m}\frac{\operatorname{Log}^r k}{k}\right) + n\sum_{j=1}^{r}\binom{r}{j}(-1)^j \operatorname{Log}^j n\left\{\sum_{k=1}^{mn}\left(\frac{\operatorname{Log}^{r-j}(k+x)}{k+x}\right.\right.$$

$$\left.\left. - \frac{\operatorname{Log}^{r-j} k}{k}\right) + \sum_{k=1}^{mn}\frac{\operatorname{Log}^{r-j} k}{k} - \frac{\operatorname{Log}^{r-j+1}(mn)}{r-j+1}\right\}$$

$$+ n\sum_{j=1}^{r}\binom{r}{j}(-1)^j \operatorname{Log}^j n\frac{\operatorname{Log}^{r-j+1}(mn)}{r-j+1}. \tag{22.3}$$

By (22.2) and Entry 22(i),

$$\lim_{m\to\infty} n\sum_{j=1}^{r}\binom{r}{j}(-1)^j \operatorname{Log}^j n\left\{\sum_{k=1}^{mn}\left(\frac{\operatorname{Log}^{r-j}(k+x)}{k+x} - \frac{\operatorname{Log}^{r-j} k}{k}\right)\right.$$

$$\left. + \sum_{k=1}^{mn}\frac{\operatorname{Log}^{r-j} k}{k} - \frac{\operatorname{Log}^{r-j+1}(mn)}{r-j+1}\right\}$$

$$= n\sum_{j=1}^{r}\binom{r}{j}(-1)^j \operatorname{Log}^j n\{-\varphi_{r-j}(x) + c_{r-j}\}. \tag{22.4}$$

We next examine the last sum on the far right side of (22.3). Replacing j by $r-j+1$, we see that this sum equals

$$\sum_{j=1}^{r}\binom{r}{j-1}\frac{(-1)^{r-j+1}\operatorname{Log}^{r-j+1} n \operatorname{Log}^{j}(mn)}{j}$$
$$=\frac{1}{r+1}\sum_{j=0}^{r}\binom{r+1}{j}(-1)^{r-j+1}\operatorname{Log}^{r-j+1} n \sum_{k=0}^{j}\binom{j}{k}\operatorname{Log}^{k} m \operatorname{Log}^{j-k} n$$
$$+\frac{(-1)^{r}}{r+1}\operatorname{Log}^{r+1} n$$
$$=(-1)^{r+1}r!\sum_{k=0}^{r}\frac{\operatorname{Log}^{k} m \operatorname{Log}^{r-k+1} n}{k!}\sum_{j=k}^{r}\frac{(-1)^{j}}{(r-j+1)!\,(j-k)!}$$
$$+\frac{(-1)^{r}}{r+1}\operatorname{Log}^{r+1} n$$
$$=(-1)^{r+1}r!\sum_{k=0}^{r}\frac{(-1)^{k}\operatorname{Log}^{k} m \operatorname{Log}^{r-k+1} n}{k!\,(r-k+1)!}\left\{\sum_{\mu=0}^{r-k+1}(-1)^{\mu}\binom{r-k+1}{\mu}\right.$$
$$\left.+(-1)^{r-k}\right\}+\frac{(-1)^{r}}{r+1}\operatorname{Log}^{r+1} n$$
$$=-r!\sum_{k=0}^{r}\frac{\operatorname{Log}^{k} m \operatorname{Log}^{r-k+1} n}{k!\,(r-k+1)!}+\frac{(-1)^{r}}{r+1}\operatorname{Log}^{r+1} n$$
$$=-\frac{1}{r+1}\operatorname{Log}^{r+1}(mn)+\frac{1}{r+1}\operatorname{Log}^{r+1} m+\frac{(-1)^{r}}{r+1}\operatorname{Log}^{r+1} n. \tag{22.5}$$

Returning now to (22.3), taking the limit as m tends to ∞, and using (22.4) and (22.5), we deduce that

$$n\varphi_{r}(x)-\sum_{j=0}^{n-1}\varphi_{r}\left(\frac{x-j}{n}\right)$$
$$=n\lim_{m\to\infty}\left\{\sum_{k=1}^{mn}\frac{\operatorname{Log}^{r} k}{k}-\sum_{k=1}^{m}\frac{\operatorname{Log}^{r} k}{k}-\frac{1}{r+1}\operatorname{Log}^{r+1}(mn)\right.$$
$$\left.+\frac{1}{r+1}\operatorname{Log}^{r+1} m\right\}+n\sum_{j=1}^{r}\binom{r}{j}(-1)^{j+1}\operatorname{Log}^{j} n\{\varphi_{r-j}(x)-c_{r-j}\}$$
$$+\frac{(-1)^{r} n}{r+1}\operatorname{Log}^{r+1} n$$
$$=n\sum_{j=1}^{r}\binom{r}{j}(-1)^{j+1}\operatorname{Log}^{j} n\{\varphi_{r-j}(x)-c_{r-j}\}+\frac{(-1)^{r} n}{r+1}\operatorname{Log}^{r+1} n,$$

by Entry 22(i). This completes the proof.

Entry 23. *Let c_r be defined as in Entry 22(i). If $r\geq 0$ and $\sigma>0$, then*

$$\sum_{k=1}^{\infty}\frac{\operatorname{Log}^{r} k}{k^{s+1}}=\frac{\Gamma(r+1)}{s^{r+1}}+\sum_{k=0}^{\infty}\frac{(-1)^{k}c_{r+k}s^{k}}{k!}. \tag{23.1}$$

Proof. In Section 13 of Chapter 7, Ramanujan gives the Laurent expansion

$$\zeta(s+1) = \frac{1}{s} + \sum_{k=0}^{\infty} \frac{(-1)^k c_k s^k}{k!}, \tag{23.2}$$

where s is any complex number. Thus, in the sequel, we shall assume that $r > 0$. (If r were restricted to the nonnegative integers, then (23.1) could be established by differentiating (23.2) r times.)

Let $f(t) = t^{-s-1} \operatorname{Log}^r t$, r, $s > 0$, in the Euler–Maclaurin summation formula (I3). We then find that

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\operatorname{Log}^r k}{k^{s+1}} &= \int_1^{\infty} \frac{\operatorname{Log}^r t}{t^{s+1}}\, dt + \int_1^{\infty} (t - [t] - \tfrac{1}{2}) f'(t)\, dt \\ &= \frac{\Gamma(r+1)}{s^{r+1}} + \int_1^{\infty} (t-[t]) f'(t)\, dt. \end{aligned}\tag{23.3}$$

Now

$$\begin{aligned}\int_1^{\infty} (t-[t]) f'(t)\, dt &= \int_1^{\infty} (t-[t]) \left(\frac{r \operatorname{Log}^{r-1} t - (s+1) \operatorname{Log}^r t}{t^{s+2}} \right) dt \\ &= \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} \int_1^{\infty} \frac{t-[t]}{t^2} (r \operatorname{Log}^{r+k-1} t - (s+1) \operatorname{Log}^{r+k} t)\, dt \\ &= \sum_{k=0}^{\infty} \frac{(-s)^k a_k}{k!}, \end{aligned}\tag{23.4}$$

where, for $k \geq 0$ and integral x,

$$a_k = \lim_{x \to \infty} \int_1^x \frac{t-[t]}{t^2} \{(r+k) \operatorname{Log}^{r+k-1} t - \operatorname{Log}^{r+k} t\}\, dt. \tag{23.5}$$

First,

$$\int_1^x \frac{(r+k) \operatorname{Log}^{r+k-1} t - \operatorname{Log}^{r+k} t}{t}\, dt = \operatorname{Log}^{r+k} x - \frac{\operatorname{Log}^{r+k+1} x}{r+k+1}. \tag{23.6}$$

Secondly,

$$\begin{aligned}&\int_1^x \frac{[t]}{t^2} \{(r+k) \operatorname{Log}^{r+k-1} t - \operatorname{Log}^{r+k} t\}\, dt \\ &\qquad = \sum_{j=1}^{x-1} j \int_j^{j+1} \frac{(r+k) \operatorname{Log}^{r+k-1} t - \operatorname{Log}^{r+k} t}{t^2}\, dt \\ &\qquad = \sum_{j=1}^{x-1} j \left\{ \frac{\operatorname{Log}^{r+k}(j+1)}{j+1} - \frac{\operatorname{Log}^{r+k} j}{j} \right\} \\ &\qquad = -\sum_{j=1}^{x} \frac{\operatorname{Log}^{r+k} j}{j} + \operatorname{Log}^{r+k} x. \end{aligned}\tag{23.7}$$

Using (23.6) and (23.7) in (23.5), we deduce that

$$a_k = \lim_{x\to\infty} \left(\sum_{j=1}^{x} \frac{\text{Log}^{r+k} j}{j} - \frac{\text{Log}^{r+k+1} x}{r+k+1} \right) = c_{r+k}, \tag{23.8}$$

by Entry 22(i). Putting (23.8) into (23.4) and then (23.4) into (23.3), we complete the proof for $s > 0$. By analytic continuation, (23.1) is valid for $\sigma > 0$.

Example 1. $-\zeta'''(\tfrac{3}{2}) = 96.001$ *nearly.*

Example 2. $-\zeta'(2) = 0.9382$ *nearly.*

Example 3. $\zeta^{(\text{iv})}(2) = 24$ *nearly.*

Example 4. $-\zeta^{(\text{v})}(\tfrac{3}{2}) = 7680$ *nearly.*

Example 5. $\sum_{k=1}^{\infty} \frac{\text{Log}^{11/2} k}{k^2} = 288$ *nearly.*

Except for the notation of the Riemann zeta-function, we have quoted Ramanujan in Examples 1–5. Ramanujan obviously used (23.1) to obtain his numerical values. As we shall see below, all of Ramanujan's approximations are quite reasonable, although he made a slight error in the calculation of $\zeta'(2)$. In fact,

$$-\zeta'(2) = 0.93754825431584375.$$

This determination was achieved by Rosser and Schoenfeld [1]. Knuth [1] and Wrench [2] later calculated $\zeta'(2)$ to 40 and 120 decimal places, respectively. It also should be noted that Gauss [1, p. 359], calculated $\zeta'(2)$ to 10 places.

In order to calculate Examples 1, 3, and 4, write, by (23.1),

$$\sum_{k=1}^{\infty} \frac{\text{Log}^r k}{k^{s+1}} = \frac{\Gamma(r+1)}{s^{r+1}} + \sum_{k=0}^{n-1} \frac{(-1)^k c_{r+k} s^k}{k!} + \sum_{k=n}^{\infty} \frac{(-1)^k c_{r+k} s^k}{k!}$$

$$= \frac{\Gamma(r+1)}{s^{r+1}} + M_n(r, s) + E_n(r, s), \tag{23.9}$$

say. We shall determine $M_n(r, s)$ numerically and estimate $E_n(r, s)$ for appropriate values of n, r, and s. Values for c_0, c_1, c_2, and c_3 were calculated by Ramanujan in Section 13 of Chapter 7. Liang and Todd [1] have calculated c_k, $0 \le k \le 19$, to 15 decimal places, and we have used their values in the calculations which follow. We also shall use the bounds,

$$\left| \frac{c_k}{k!} \right| \le \frac{4}{k\pi^k}, \qquad k \ge 1, \tag{23.10}$$

found in Berndt's paper [1, Theorem 2].

Examining Example 1, we set $r=3$ and $s=\frac{1}{2}$ in (23.9) and use (23.10) to find that

$$|E_n(3,\tfrac{1}{2})| \le \frac{4}{2^n\pi^{n+3}}\sum_{j=0}^{\infty}\frac{(n+2+j)(n+1+j)}{(2\pi)^j}$$
$$=\frac{4}{2^n\pi^{n+3}}\left\{\frac{(n+2)(n+1)}{1-1/(2\pi)}+\frac{2n+4}{(1-1/(2\pi))^2}+\frac{2}{(1-1/(2\pi))^3}\right\}.$$

Letting $n=6$, we find that $|E_6(3,\frac{1}{2})| \le 0.000195$. Using the calculations of Liang and Todd [1], we find that $M_6(3,\frac{1}{2})=0.009940$. In summary, we have shown that $-\zeta'''(\frac{3}{2})=96.009940$ with an error no greater than 0.000195. Thus, 96.01 would have been a better approximation than 96.001. Ramanujan probably found his approximation by taking $n=2$ in (23.9), for it turns out that $M_2(3,\frac{1}{2})=0.000891$.

To calculate $\zeta^{(\text{iv})}(2)$ we want to put $r=4$ and $s=1$ in (23.9). Using (23.10), we determine that

$$|E_n(4,1)| = \frac{4}{\pi^{n+4}}\sum_{j=0}^{\infty}\frac{(n+3+j)(n+2+j)(n+1+j)}{\pi^j}.$$

In particular, if $n=5$ we find that $|E_5(4,1)| \le 0.008670$. Using Liang and Todd's [1] calculations, we deduce that $\zeta^{(\text{iv})}(2)=24.014859$ with an error of at most 0.008670. This justifies Ramanujan's claim.

For Example 4, we put $r=5$, $s=\frac{1}{2}$, and $n=4$ in (23.9). Proceeding as above, we find that $-\zeta^{(\text{v})}(\frac{3}{2})=7680.008541$ with an error no more than 0.045880, and so Ramanujan's approximation is again corroborated.

For nonintegral r, c_{r+k} has not been calculated in the literature for any values of k. Moreover, no estimates like those in (23.10) have been determined. Thus, we do not give a careful error analysis for Example 5. We merely note that $\Gamma(\frac{13}{2})=287.88528$ (Abramowitz and Stegun [1, p. 272]), and so Ramanujan's approximation of 288 appears to be a reasonable one.

In Section 24, the last section of Chapter 8, Ramanujan studies

$$\varphi(x)=\sum_{k=1}^{x}\frac{\operatorname{Log} k}{\sqrt{k}}, \tag{24.1}$$

which is the case $r=\frac{1}{2}$ of (21.1). Entry 24(i) is the equality

$$\varphi(x)=\sum_{k=1}^{\infty}\left\{\frac{\operatorname{Log} k}{\sqrt{k}}-\frac{\operatorname{Log}(k+x)}{\sqrt{k+x}}\right\}, \tag{24.2}$$

which really is the definition of $\varphi(x)$ for all values of x, $x>-1$, and is the case $r=\frac{1}{2}$ of (21.10).

Entry 24(ii). *Let* $\varphi(x)$ *be defined by* (24.1). *Then as* x *tends to* ∞,

$$\varphi(x)-\left\{\sum_{k=1}^{x}\frac{1}{\sqrt{k}}-\zeta(\tfrac{1}{2})\right\}\operatorname{Log} x \sim -4\sqrt{x}-\zeta(\tfrac{1}{2})\{\tfrac{1}{2}\gamma+\tfrac{1}{4}\pi+\tfrac{1}{2}\operatorname{Log}(8\pi)\}$$
$$+\sum_{k=1}^{\infty}\frac{B_{2k}(\frac{1}{2})_{2k-1}H_{2k-2}(\frac{1}{2})}{(2k)!\,x^{2k-1/2}},$$

where $H_n(\frac{1}{2})$ *is defined by* (21.2).

Proof. Although it is assumed that $r > 1$ in Entry 21(i), that part of the proof through (21.8) is valid for $r > 0$. Thus, putting $r = \frac{1}{2}$ in (21.8), we find that, as x tends to ∞,

$$\varphi(x) \sim c + \tfrac{1}{2}x^{-1/2} \operatorname{Log} x + 2\sqrt{x} \operatorname{Log} x - 4\sqrt{x}$$
$$- \operatorname{Log} x \sum_{k=1}^{\infty} \frac{B_{2k}(\frac{1}{2})_{2k-1}}{(2k)!\, x^{2k-1/2}} + \sum_{k=1}^{\infty} \frac{B_{2k}(\frac{1}{2})_{2k-1} H_{2k-2}(\frac{1}{2})}{(2k)!\, x^{2k-1/2}}. \qquad (24.3)$$

Now by Entry 1 of Chapter 7,

$$\sum_{k=1}^{x} \frac{1}{\sqrt{k}} \sim 2\sqrt{x} + \frac{1}{2\sqrt{x}} + \zeta(\tfrac{1}{2}) - \sum_{k=1}^{\infty} \frac{B_{2k}(\frac{1}{2})_{2k-1}}{(2k)!\, x^{2k-1/2}},$$

as x tends to ∞. Thus, from (24.3),

$$\varphi(x) - \left\{\sum_{k=1}^{x} \frac{1}{\sqrt{k}} - \zeta(\tfrac{1}{2})\right\} \operatorname{Log} x \sim c - 4\sqrt{x} + \sum_{k=1}^{\infty} \frac{B_{2k}(\frac{1}{2})_{2k-1} H_{2k-2}(\frac{1}{2})}{(2k)!\, x^{2k-1/2}},$$

as x tends to ∞. It remains to show that

$$c = -\zeta(\tfrac{1}{2})\{\tfrac{1}{2}\gamma + \tfrac{1}{4}\pi + \tfrac{1}{2} \operatorname{Log}(8\pi)\}. \qquad (24.4)$$

It is clear from (24.3) that

$$c = \lim_{x \to \infty} \{\varphi(x) - 2\sqrt{x} \operatorname{Log} x + 4\sqrt{x}\}. \qquad (24.5)$$

We shall first show that $c = -\zeta'(\frac{1}{2})$. From (19.18),

$$\zeta'(\tfrac{1}{2}) = \int_1^{\infty} \frac{[x] - x + \frac{1}{2}}{x^{3/2}} - \int_1^{\infty} \frac{[x] - x + \frac{1}{2}}{2x^{3/2}} \operatorname{Log} x\, dx - 4$$
$$= \lim_{n \to \infty} \int_1^n ([x] - x + \tfrac{1}{2}) \frac{d}{dx}\left(\frac{\operatorname{Log} x}{\sqrt{x}}\right) dx - 4$$
$$= \lim_{n \to \infty} \left(\sum_{k=1}^{n-1} k \int_k^{k+1} \frac{d}{dx}\left(\frac{\operatorname{Log} x}{\sqrt{x}}\right) dx\right.$$
$$\left. - (x - \tfrac{1}{2}) \frac{\operatorname{Log} x}{\sqrt{x}}\bigg|_1^n + \int_1^n \frac{\operatorname{Log} x}{\sqrt{x}}\, dx\right) - 4$$
$$= \lim_{n \to \infty} \left(\sum_{k=1}^{n-1} k \left\{\frac{\operatorname{Log}(k+1)}{\sqrt{k+1}} - \frac{\operatorname{Log} k}{\sqrt{k}}\right\} + \sqrt{n} \operatorname{Log} n - 4\sqrt{n}\right)$$
$$= \lim_{n \to \infty} \left(-\sum_{k=1}^{n} \frac{\operatorname{Log} k}{\sqrt{k}} + 2\sqrt{n} \operatorname{Log} n - 4\sqrt{n}\right) = -c, \qquad (24.6)$$

by (24.5).

From Titchmarsh's book [3, p. 20],

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} = -\operatorname{Log}(2\pi) - \tfrac{1}{2}\pi \tan(\tfrac{1}{2}\pi s) + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)},$$

Setting $s=\frac{1}{2}$, using Corollary 3(i) in Section 6, and using (24.6), we establish (24.4). This completes the proof of Entry 24(ii).

For Entries 24(iii)–(v), we shall define

$$\psi(x)=\sum_{k=1}^{\infty}\left\{\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+x}}\right\}, \qquad x>-1. \tag{24.7}$$

Entry 24(iii). *Let* $\varphi(x)$ *and* $\psi(x)$ *be defined by* (24.2) *and* (24.7), *respectively, for* $x>-1$. *If n is any positive integer, then*

$$\varphi(x)-\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\varphi\left(\frac{x-j}{n}\right)=\psi(x)\operatorname{Log} n-(\sqrt{n}-1)c-\zeta(\tfrac{1}{2})\operatorname{Log} n,$$

where c is given by (24.4).

Proof. For $\sigma>1$, the calculation (21.14) gives

$$\sum_{k=1}^{\infty}\left(\left\{\frac{\operatorname{Log} k}{k^s}-\frac{\operatorname{Log}(k+x)}{(k+x)^s}\right\}-\frac{1}{n^s}\sum_{j=0}^{n-1}\left\{\frac{\operatorname{Log} k}{k^s}-\frac{\operatorname{Log}\{(kn-j+x)/n\}}{\{(kn-j+x)/n\}^s}\right\}\right)$$
$$=(n^{1-s}-1)\zeta'(s)+\operatorname{Log} n\sum_{k=1}^{\infty}\left\{\frac{1}{k^s}-\frac{1}{(k+x)^s}\right\}-\operatorname{Log} n\,\zeta(s). \tag{24.8}$$

By analytic continuation, the extremal sides of (24.8) are equal for $\sigma>0$. Now set $s=\frac{1}{2}$ in (24.8) and use (24.2), (24.6), and (24.7) to obtain the desired equality.

Entry 24(iv). *Let* $\varphi(x)$ *and* $\psi(x)$ *be defined by* (24.2) *and* (24.7), *respectively. If* $0<x<1$, *then*

$$\varphi(x-1)+\varphi(-x)-2c+(\gamma+\tfrac{1}{2}\pi+\operatorname{Log}(8\pi))\{\psi(x-1)+\psi(-x)-2\zeta(\tfrac{1}{2})\}$$
$$=2\sum_{k=1}^{\infty}\frac{\operatorname{Log} k\cos(2\pi kx)}{\sqrt{k}},$$

where c is given in (24.4).

Proof. For $\sigma>0$, define

$$F(s)=\sum_{k=1}^{\infty}\left\{\frac{1}{(k+x-1)^s}+\frac{1}{(k-x)^s}-\frac{2}{k^s}\right\}.$$

Since $\zeta(s)$ and $\zeta(s,a)$ can be analytically continued into the entire complex s-plane, $F(s)$ can be so continued. In view of (24.7), we note that

$$F(\tfrac{1}{2})=-\{\psi(x-1)+\psi(-x)\}, \tag{24.9}$$

and by (24.2), we observe that

$$F'(\tfrac{1}{2})=\varphi(x-1)+\varphi(-x). \tag{24.10}$$

Now apply the functional equation (18.8) of $\zeta(s)$ and Hurwitz's formula (17.15) for $\zeta(s, a)$ to obtain

$$F(s) = \frac{4\Gamma(1-s)}{(2\pi)^{1-s}} \sin(\tfrac{1}{2}\pi s)\left\{\sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{k^{1-s}} - \zeta(1-s)\right\}, \tag{24.11}$$

where $0 < x < 1$ and $\sigma < 1$. Differentiating (24.11), we find that

$$\begin{aligned} F'(s) = {} & \left\{-\frac{4\Gamma'(1-s)}{(2\pi)^{1-s}} \sin(\tfrac{1}{2}\pi s) + (2\pi)^s\Gamma(1-s)\cos(\tfrac{1}{2}\pi s)\right\} \\ & \cdot \left\{\sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{k^{1-s}} - \zeta(1-s)\right\} \\ & + \frac{4\Gamma(1-s)}{(2\pi)^{1-s}} \sin(\tfrac{1}{2}\pi s)\left\{\sum_{k=1}^{\infty} \frac{\operatorname{Log}(2\pi k)\cos(2\pi kx)}{k^{1-s}} \right. \\ & \left. - \operatorname{Log}(2\pi)\zeta(1-s) + \zeta'(1-s)\right\}. \end{aligned} \tag{24.12}$$

Using (24.9), (24.10), (24.11), and (24.12) with $s = \frac{1}{2}$, Corollary 3(i) in Section 6, and (24.4), we find that

$$\begin{aligned} & \varphi(x-1) + \varphi(-x) - 2c + (\gamma + \tfrac{1}{2}\pi + \operatorname{Log}(8\pi))\{\psi(x-1) + \psi(-x) - 2\zeta(\tfrac{1}{2})\} \\ & \quad = F'(\tfrac{1}{2}) - 2c - (\gamma + \tfrac{1}{2}\pi + \operatorname{Log}(8\pi))\{F(\tfrac{1}{2}) + 2\zeta(\tfrac{1}{2})\} \\ & \quad = (2\gamma + 4\operatorname{Log} 2 + \pi)\left\{\sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{\sqrt{k}} - \zeta(\tfrac{1}{2})\right\} \\ & \qquad + 2\left\{\sum_{k=1}^{\infty} \frac{\operatorname{Log}(2\pi k)\cos(2\pi kx)}{\sqrt{k}} - \operatorname{Log}(2\pi)\zeta(\tfrac{1}{2}) + \zeta'(\tfrac{1}{2})\right\} \\ & \qquad - 2c - 2(\gamma + \tfrac{1}{2}\pi + \operatorname{Log}(8\pi)) \sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{\sqrt{k}} \\ & \quad = 2\sum_{k=1}^{\infty} \frac{\operatorname{Log} k \cos(2\pi kx)}{\sqrt{k}}. \end{aligned}$$

Entry 24(v). *Under the same hypotheses as Entry* 24(iv),

$$\varphi(x-1) - \varphi(-x) + (\gamma - \tfrac{1}{2}\pi + \operatorname{Log}(8\pi))\{\psi(x-1) - \psi(-x)\} = 2\sum_{k=1}^{\infty} \frac{\operatorname{Log} k \sin(2\pi kx)}{\sqrt{k}}.$$

Proof. For $\sigma > 0$, define

$$G(s) = \sum_{k=1}^{\infty} \left\{\frac{1}{(k-x)^s} - \frac{1}{(k+x-1)^s}\right\}.$$

As with F in the previous proof, G can be analytically continued into the entire complex s-plane. Note that

$$G(\tfrac{1}{2}) = \psi(x-1) - \psi(-x)$$

and

$$G'(\tfrac{1}{2}) = -\varphi(x-1) - \varphi(-x).$$

By Hurwitz's formula (17.15),

$$G(s) = -\frac{4\Gamma(1-s)}{(2\pi)^{1-s}} \cos(\tfrac{1}{2}\pi s) \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k^{1-s}},$$

where $0 < x < 1$ and $\sigma < 1$. Proceeding as in the previous proof, we deduce that

$$\begin{aligned}
&\varphi(x-1) - \varphi(-x) + (\gamma - \tfrac{1}{2}\pi + \operatorname{Log}(8\pi))\{\psi(x-1) - \psi(-x)\} \\
&\quad = -G'(\tfrac{1}{2}) + (\gamma - \tfrac{1}{2}\pi + \operatorname{Log}(8\pi))G(\tfrac{1}{2}) \\
&\quad = (2\gamma + 4\operatorname{Log} 2 - \pi) \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{\sqrt{k}} + 2 \sum_{k=1}^{\infty} \frac{\operatorname{Log}(2\pi k)\sin(2\pi k x)}{\sqrt{k}} \\
&\qquad - 2(\gamma - \tfrac{1}{2}\pi + \operatorname{Log}(8\pi)) \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{\sqrt{k}} \\
&\quad = 2 \sum_{k=1}^{\infty} \frac{\operatorname{Log} k \sin(2\pi k x)}{\sqrt{k}}.
\end{aligned}$$

In Example 1, Ramanujan asks us to "Find the values of $\varphi(-\tfrac{1}{2})$, $\varphi(-\tfrac{2}{3})$, & $\varphi(-\tfrac{3}{4})$," but he does not record these values. Putting $n = 2$ and $x = 0$ in Entry 24(iii), we readily find that

$$\varphi(-\tfrac{1}{2}) = (\sqrt{2} - 2)(\tfrac{1}{2}\gamma + \tfrac{1}{4}\pi + \tfrac{1}{2}\operatorname{Log}(8\pi))\zeta(\tfrac{1}{2}) + \sqrt{2}\operatorname{Log} 2\, \zeta(\tfrac{1}{2}).$$

To calculate $\varphi(-\tfrac{2}{3})$, first put $x = 0$ and $n = 3$ in Entry 24(iii) to obtain a formula for $\varphi(-\tfrac{1}{3}) + \varphi(-\tfrac{2}{3})$. Next, put $x = \tfrac{1}{3}$ in Entry 24(v) to obtain a formula for $\varphi(-\tfrac{2}{3}) - \varphi(-\tfrac{1}{3})$. From these two formulas, one can obtain a formula for $\varphi(-\tfrac{2}{3})$, but it is not very elegant or enlightening, and so we do not record it. Similarly, we can derive a formula for $\varphi(-\tfrac{3}{4})$.

Example 2(a). *If c is defined by* (24.4), *then* $c = 3.92265$.

Using the value $\zeta(\tfrac{1}{2}) = -1.4603545088$ found in Dwight's tables [1, p. 244], we can easily verify that Ramanujan's calculated value of c is correct to the number of places given.

Example 2(b). *We have*

$$c \sim 4 - \sum_{k=1}^{\infty} \frac{B_{2k}(\tfrac{1}{2})_{2k-1} H_{2k-2}(\tfrac{1}{2})}{(2k)!}. \tag{24.13}$$

Formula (24.13) can be formally derived by setting $x = 1$ and $r = \frac{1}{2}$ in (21.3). The infinite series on the right side of (24.13) is semiconvergent in the sense that the error made in terminating the series at the nth term is less in modulus than the modulus of the $(n + 1)$th term. (See Bromwich's book [1, p. 328].) It is with this understanding that we interpret the sign $\sim$ in (24.13). We remark that (24.13) apparently cannot be used to determine c as accurately as in Example 2(a).

CHAPTER 9

Infinite Series Identities, Transformations, and Evaluations

Chapter 9 fully illustrates Hardy's declaration in Ramanujan's *Collected Papers* [15, p. xxxv], "It was his insight into algebraical formulae, transformations of infinite series, and so forth, that was most amazing." This chapter has 35 sections containing 139 formulas of which many are, indeed, very beautiful and elegant. Ramanujan gives several transformations of power series leading to many striking series relations and attractive series evaluations. Most of Ramanujan's initial efforts in this direction pertain to the dilogarithm and related functions. As is to be expected, these results are not new and can be traced back to Euler, Landen, Abel, and others. However, most of Ramanujan's remaining findings on transformations of power series appear to be new.

The beautiful formula

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k!)^2}{(2k)!\, k^3} \tag{0.1}$$

has been made famous by Apéry's proof of the irrationality of $\zeta(3)$. For example, see papers of Apéry [1], Cohen [1], Mendès France [1], Reyssat [1], and van der Poorten [1]. Ramanujan evidently missed this formula, but Chapter 9 contains several intriguing formulas of the same type. Some of these involve $\zeta(3)$ and Catalan's constant.

In Chapter 8, Ramanujan studied certain functions which are akin to $\operatorname{Log}\Gamma(x+1)$. In Sections 27–30 of Chapter 9, Ramanujan returns to this topic. The generalization studied here is very closely related to that studied by Bendersky [1] and more recently by Büsing [1]. Except for a simple result in Section 31, the material in Sections 27–30 has no relation to the rest of Chapter 9.

In analyzing Ramanujan's work, Hardy has frequently pointed out that "he knew no theory of functions" [20, p. 14]. Many of the formulas in Chapter 9 can be extended by analytic continuation to complex values of x. However, because Ramanujan obviously intended his results to hold for just real values of x, we have presented his theorems in this more restricted setting. We have made exceptions to this decision in the few instances when vacuous theorems would otherwise result.

Several of Ramanujan's formulas in Chapter 9 need minor corrections. However, there are a few results, for example, Entry 3 and formula (11.3), which are evidently quite wrong. In describing three beautiful formulas of Ramanujan which he could not prove, Hardy [20, p. 9] has written, "They must be true because, if they were not true, no one would have had the imagination to invent them." Although this judgement seems invalid here, Hardy may be correct, because very likely "corrected versions" of Ramanujan's incorrect formulas exist. Unfortunately, we have no insights as to what these "corrected versions" might be.

Entry 1. *For each positive integer r, define*

$$S_r = \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k+1-a)^r} - \frac{1}{(2k+1+a)^r} \right\},$$

where a is real but not an odd integer. Assume that $|x| < \pi$. *Then if r is an odd positive integer,*

(i)
$$c_r(x) \equiv \sum_{k=0}^{\infty} \left\{ \frac{\cos(2k+1-a)x}{(2k+1-a)^r} - \frac{\cos(2k+1+a)x}{(2k+1+a)^r} \right\} = \sum_{k=0}^{(r-1)/2} \frac{(-1)^k S_{r-2k} x^{2k}}{(2k)!},$$

while if r is an even positive integer,

(ii)
$$s_r(x) \equiv \sum_{k=0}^{\infty} \left\{ \frac{\sin(2k+1-a)x}{(2k+1-a)^r} - \frac{\sin(2k+1+a)x}{(2k+1+a)^r} \right\} = \sum_{k=0}^{r/2-1} \frac{(-1)^k S_{r-2k-1} x^{2k+1}}{(2k+1)!}.$$

Proof. We first establish (i) for $r = 1$. Observe that

$$c_1(x) = 2a\cos(ax) \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2 - a^2} + 2\sin(ax) \sum_{k=0}^{\infty} \frac{(2k+1)\sin(2k+1)x}{(2k+1)^2 - a^2}. \tag{1.1}$$

By a well-known formula found in Bromwich's text [1, p. 368], for $|x| < \pi$,

$$\frac{\pi}{2a} \frac{\cos(ax)}{\sin(a\pi)} = \frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(nx)}{n^2 - a^2}, \tag{1.2}$$

from which it follows that, if $0 < x < \pi$,

$$\frac{\pi}{2a}\frac{\cos a(x-\pi)}{\sin(a\pi)} = \frac{1}{2a^2} - \sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2-a^2}. \tag{1.3}$$

Subtracting (1.3) from (1.2), we get, for $0 < x < \pi$,

$$2\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2-a^2} = \frac{\pi}{2a\sin(a\pi)}\{\cos(ax) - \cos a(x-\pi)\}. \tag{1.4}$$

Differentiating both sides of (1.2) with respect to x and proceeding as above, we find that, for $0 < x < \pi$,

$$2\sum_{k=0}^{\infty}\frac{(2k+1)\sin(2k+1)x}{(2k+1)^2-a^2} = \frac{\pi}{2\sin(a\pi)}\{\sin(ax) - \sin a(x-\pi)\}. \tag{1.5}$$

Substituting (1.4) and (1.5) into (1.1), we deduce that, for $0 < x < \pi$,

$$\begin{aligned} c_1(x) &= \frac{\pi}{2\sin(a\pi)}(\cos(ax)\{\cos(ax) - \cos a(x-\pi)\} \\ &\quad + \sin(ax)\{\sin(ax) - \sin a(x-\pi)\}) \\ &= \frac{\pi}{2}\tan\left(\frac{a\pi}{2}\right) \\ &= 2a\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2-a^2} = S_1. \end{aligned} \tag{1.6}$$

Trivially, (1.6) holds for $x = 0$. Replacing x by $-x$, we see that (1.6) has thus been proven for $|x| < \pi$. This completes the proof of (i) for $r = 1$.

If we integrate the extremal sides of (1.6) over $[0, x]$, $|x| < \pi$, we readily find that

$$s_2(x) = S_1 x,$$

which is in agreement with (ii) when $r = 2$.

We now proceed by induction. Assume that (i) is valid for some positive odd integer r. Integrating both sides of (i) over $[0, x]$, $|x| < \pi$, we find that,

$$s_{r+1}(x) = \sum_{k=0}^{(r-1)/2}\frac{(-1)^k S_{r-2k}x^{2k+1}}{(2k+1)!}, \tag{1.7}$$

which is precisely (ii) with r replaced by $r+1$. Integrating (1.7) over $[0, x]$, $|x| < \pi$, we find that

$$-c_{r+2}(x) + S_{r+2} = \sum_{k=0}^{(r-1)/2}\frac{(-1)^k S_{r-2k}x^{2k+2}}{(2k+2)!},$$

or

$$c_{r+2}(x) = \sum_{k=0}^{(r+1)/2}\frac{(-1)^k S_{r+2-2k}x^{2k}}{(2k)!},$$

which is (i) with r replaced by $r+2$. This completes the proofs of both (i) and (ii).

Entry 2. *For each integer r with $r \geq 2$, define*

$$S_r = \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k+1-a)^r} + \frac{1}{(2k+1+a)^r} \right\},$$

where a is real but not an odd integer. Assume that $0 < x < \pi$. Then if r is an even positive integer,

(i)
$$\begin{aligned} c_r(x) &\equiv \sum_{k=0}^{\infty} \left\{ \frac{\cos(2k+1-a)x}{(2k+1-a)^r} + \frac{\cos(2k+1+a)x}{(2k+1+a)^r} \right\} \\ &= \sum_{k=0}^{r/2-1} \frac{(-1)^k S_{r-2k} x^{2k}}{(2k)!} + \frac{(-1)^{r/2} \pi x^{r-1}}{2(r-1)!}, \end{aligned}$$

and if r is an odd positive integer,

(ii)
$$\begin{aligned} s_r(x) &\equiv \sum_{k=0}^{\infty} \left\{ \frac{\sin(2k+1-a)x}{(2k+1-a)^r} + \frac{\sin(2k+1+a)x}{(2k+1+a)^r} \right\} \\ &= \sum_{k=0}^{(r-3)/2} \frac{(-1)^k S_{r-1-2k} x^{2k+1}}{(2k+1)!} + \frac{(-1)^{(r-1)/2} \pi x^{r-1}}{2(r-1)!}. \end{aligned}$$

Proof. We first establish (ii) for $r = 1$. Using (1.4) and (1.5), we find that, for $0 < x < \pi$,

$$\begin{aligned} s_1(x) &= 2\cos(ax) \sum_{k=0}^{\infty} \frac{(2k+1)\sin(2k+1)x}{(2k+1)^2 - a^2} \\ &\quad - 2a\sin(ax) \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2 - a^2} \\ &= \frac{\pi}{2\sin(a\pi)} (\cos(ax)\{\sin(ax) - \sin a(x-\pi)\} \\ &\quad - \sin(ax)\{\cos(ax) - \cos a(x-\pi)\}) \\ &= \frac{\pi}{2}, \end{aligned} \tag{2.1}$$

which proves (ii) when $r = 1$.

Integrating both sides of (2.1) over $[0, x]$, $0 < x < \pi$, we find that

$$-c_2(x) + S_2 = \frac{\pi x}{2},$$

which establishes (i) for $r = 2$.

Proceeding by induction on r, we assume that (i) is valid for an arbitrary even positive integer r. Integrating (i) over $[0, x]$, $0 < x < \pi$, we readily achieve (ii) with r replaced by $r+1$. A second integration yields (i) with r replaced by $r+2$. Since the details are like those in the previous proof, we omit them. This completes the induction.

Ramanujan, p. 104, supplies the following incomplete hint for his apparently invalid argument: "In both 1 & 2 expand the series in ascending powers of x and apply."

The last term on the right sides of both (i) and (ii) in Entry 2 is absent in the notebooks.

In preparation for Ramanujan's next formula, we make some definitions. Let

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

$$S_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \sin(2k+1)x}{(2k+1)^n}, \qquad C_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cos(2k+1)x}{(2k+1)^n},$$

and

$$\varphi_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k H_{k+1} \cos(2k+1)x}{(2k+1)^n}, \tag{3.1}$$

where n is any natural number and x is real. Unfortunately, Entry 3 is false for at least n sufficiently large. We are unable to offer a corrected version of Ramanujan's formula. It appears that if a corrected formula exists, its shape would be significantly different.

Entry 3. *Let S_n, C_n, and φ_n be defined as above. If n is an odd integer at least equal to* 3, *then*

$$\varphi_{n-2}(x) - \varphi_n(x) = xS_{n-2}(x) - xS_n(x) + nC_{n-1}(x) - nC_{n+1}(x). \tag{3.2}$$

Disproof (*for n sufficiently large*). First, observe that (3.2) is certainly false for all n if x is any odd multiple of $\pi/2$.

If x is not an odd multiple of $\pi/2$, then a brief calculation shows that

$$\varphi_{n-2}(x) - \varphi_n(x) \sim -\frac{4\cos(3x)}{3^{n-1}}, \tag{3.3}$$

as n tends to ∞. On the other hand, a similar argument shows that

$$xS_{n-2}(x) - xS_n(x) + nC_{n-1}(x) - nC_{n+1}(x) \sim -\frac{8x\sin(3x)}{3^n} - \frac{8n\cos(3x)}{3^{n+1}}, \tag{3.4}$$

as n tends to ∞. For large n, (3.3) and (3.4) are incompatible.

In order to state Entries 4(i) and (ii), we need to make several definitions. For each nonnegative integer n, define

$$F_n(x) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k \sin(2k+1)x}{2^{2k}(2k+1)^n} - (-1)^n \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k \sin(2k+2)x}{2^{2k}(2k+2)^n}$$
$$+ (-1)^n \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k \sin(2k+2)x}{2^{2k}(2k+2)^{n+1}},$$

$$\psi_n(x) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k \cos(2k+1)x}{2^{2k}(2k+1)^n} + (-1)^n \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k \cos(2k+2)x}{2^{2k}(2k+2)^n}$$
$$- (-1)^n \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k \cos(2k+2)x}{2^{2k}(2k+2)^{n+1}},$$

$$\varphi(n) = \frac{2}{\pi} \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{2k}(2k+1)^{n+1}},$$

and

$$S_n = (1 - 2^{1-n})\zeta(n),$$

where ζ denotes the Riemann zeta-function.

Entries 4(i), (ii). *Let $|x| < \pi/2$ and let n be an integer. If $n \geq 0$, then*

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \sum_{j=0}^{k} 2^{-2j} S_{2j} \varphi(2k-2j)$$
$$= \begin{cases} \frac{1}{2} \sin(\pi n/2) F_n(x), & \text{if } n \text{ is odd,} \\ \frac{1}{2} \cos(\pi n/2) \psi_n(x), & \text{if } n \text{ is even.} \end{cases} \tag{4.1}$$

If $n \geq -1$, then

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \sum_{j=0}^{k} 2^{-2j} S_{2j} \varphi(2k+1-2j)$$
$$= \begin{cases} \frac{1}{2} \sin(\pi n/2) F_{n+1}(x), & \text{if } n \text{ is odd,} \\ \frac{1}{2} \cos(\pi n/2) \psi_{n+1}(x), & \text{if } n \text{ is even.} \end{cases} \tag{4.2}$$

Before commencing a proof of (4.1) and (4.2), we offer a few comments. Ramanujan further defines

$$\frac{2}{\pi} A_n = \left(\frac{\pi}{2}\right)^n + \left(\frac{3\pi}{2}\right)^n + \left(\frac{5\pi}{2}\right)^n + \cdots + \left(x - \frac{\pi}{2}\right)^n,$$

where evidently x is meant to be a positive multiple of π. Ramanujan's versions of Entries 4(i) and (ii) also contain formulas for $F_n(x)$ and $\psi_n(x)$ in terms of A_k, $k \leq n-1$. Because (4.1) and (4.2) implicitly indicate that x is a continuous variable, with $|x| < \pi/2$, and because A_k is defined for only values of x that are positive integral multiples of π, Ramanujan's formulas involving

A_k appear to have no sensible interpretation, and we shall not make any further comments about these formulas.

In Ramanujan's second published paper [5], [15, pp. 15–17], he establishes a recursive formula for $\varphi(n)$ in terms of S_k, $1 \le k \le n$. This recursion is also given by Ramanujan in Chapter 10, Section 13.

Proof of Entries 4(i), (ii). We proceed by induction. We first establish (4.1) for $n = 0$. If L denotes the left side of (4.1) when $n = 0$, then $L = S_0\varphi(0)$. Since $\zeta(0) = -\frac{1}{2}$, we find that $S_0 = \frac{1}{2}$. Now in Proposition 4(vii) below, which actually holds for $|x| \le \pi/2$, set $x = \pi/2$ to deduce that $\varphi(0) = 1$. Hence, $L = \frac{1}{2}$. On the other hand, by Propositions 4(ii1), (iv), and (vi) below,

$$\tfrac{1}{2}\psi_0(x) = \frac{1}{2}\left\{\frac{\cos(x/2)}{\sqrt{2\cos x}} + \frac{\cos(3x/2)}{\sqrt{2\cos x}} - \cos\left(\frac{x}{2}\right)\sqrt{2\cos x} + 1\right\} = \tfrac{1}{2},$$

and so (4.1) is valid for $n = 0$. (Propositions 4(v), (vii), and (ix) below can be used to provide a direct proof of (4.1) when $n = 1$.)

We now prove (4.2) for $n = -1$. In this case, the left side of (4.2) is understood to be equal to 0. On the other hand, by Propositions 4(i1), (iii), and (v),

$$-\tfrac{1}{2}F_0(x) = -\frac{1}{2}\left\{\frac{\sin(x/2)}{\sqrt{2\cos x}} - \frac{\sin(3x/2)}{\sqrt{2\cos x}} + \sin\left(\frac{x}{2}\right)\sqrt{2\cos x}\right\} = 0,$$

as desired.

We also prove (4.2) for $n = 0$. In this case, the left side of (4.2) is equal to

$$S_0\varphi(1) = \frac{1}{\pi}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}(2k+1)^2} = \tfrac{1}{2}\operatorname{Log} 2,$$

by Example (i) in Section 16. On the other hand, by Propositions 4(vi), (viii), and (x),

$$\begin{aligned}\tfrac{1}{2}\psi_1(x) = \tfrac{1}{2}\Big\{&\operatorname{Log}\left(\sqrt{\cos x} + \sqrt{2}\cos\left(\frac{x}{2}\right)\right) - \cos\left(\frac{x}{2}\right)\sqrt{2\cos x} + 1\\ &+ \cos\left(\frac{x}{2}\right)\sqrt{2\cos x} - \operatorname{Log}\left(\sqrt{\cos x} + \sqrt{2}\cos\left(\frac{x}{2}\right)\right) + \operatorname{Log} 2 - 1\Big\}\\ = \tfrac{1}{2}&\operatorname{Log} 2,\end{aligned}$$

and so (4.2) is valid for $n = 0$.

Proceeding by induction, we now assume that (4.1) and (4.2) are valid for any fixed, nonnegative *even* integer n. Integrating (4.1) over $[0, x]$, $|x| < \pi/2$, we readily find that

$$\sum_{k=0}^{n/2}\frac{(-1)^k x^{n+1-2k}}{(n+1-2k)!}\sum_{j=0}^{k}2^{-2j}S_{2j}\varphi(2k-2j) = \tfrac{1}{2}\sin\left(\frac{(n+1)\pi}{2}\right)F_{n+1}(x). \quad (4.3)$$

Thus, we have established (4.1) with n replaced by $n+1$.

Integrating (4.3) over $[0, x]$, $|x| < \pi/2$, we find that

$$\sum_{k=0}^{n/2} \frac{(-1)^k x^{n+2-2k}}{(n+2-2k)!} \sum_{j=0}^{k} 2^{-2j} S_{2j} \varphi(2k-2j)$$
$$= \tfrac{1}{2}\cos\left(\frac{(n+2)\pi}{2}\right)\psi_{n+2}(x) + \tfrac{1}{2}\cos\left(\frac{n\pi}{2}\right)\left\{\sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+1)^{n+2}}\right.$$
$$\left. + \sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+2)^{n+2}} - \sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+2)^{n+3}}\right\}. \qquad (4.4)$$

Comparing (4.4) with (4.1), with n replaced by $n+2$, we find that we must show that

$$\sum_{j=0}^{n/2+1} 2^{-2j} S_{2j} \varphi(n+2-2j)$$
$$= \tfrac{1}{2}\left\{\sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+1)^{n+2}} + \sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+2)^{n+2}}\right.$$
$$\left. - \sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+2)^{n+3}}\right\}. \qquad (4.5)$$

In a similar fashion, after two integrations of (4.2), we find that it suffices to show that

$$\sum_{j=0}^{n/2+1} 2^{-2j} S_{2j} \varphi(n+3-2j)$$
$$= \frac{1}{2}\left\{\sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+1)^{n+3}} - \sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+2)^{n+3}}\right.$$
$$\left. + \sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k+1)^{n+4}}\right\}. \qquad (4.6)$$

Combining (4.5) and (4.6) together, we deduce that, in order to prove Entries 4(i), (ii), it suffices to prove the following curious theorem.

Theorem. *Let n denote a nonnegative integer. Then*

$$\sum_{\substack{j=0 \\ 2|j}}^{n} 2^{-j} S_j \varphi(n-j) = \frac{1}{2}\sum_{k=0}^{\infty}\binom{2k}{k}(-4)^{-k}\{(2k+1)^{-n}$$
$$+ (-1)^n(2k+2)^{-n} - (-1)^n(2k+2)^{-n-1}\}. \qquad (4.7)$$

We are very grateful to R. J. Evans for providing the following elegant proof of (4.7).

Proof. Let L_n and R_n denote, respectively, the left and right sides of (4.7).

Define

$$L(x)=\sum_{n=0}^{\infty} L_n x^n \quad \text{and} \quad R(x)=\sum_{n=0}^{\infty} R_n x^n,$$

where $|x|$ is sufficiently small. It then suffices to show that

$$L(x)=R(x). \tag{4.8}$$

Next, define, for $j \geq 0$,

$$T_j=\begin{cases} 2^{-j}S_j, & \text{if } 2|j, \\ 0, & \text{if } 2\not| j. \end{cases} \tag{4.9}$$

Then, by (4.9) and the definition of L_n,

$$L_n=\sum_{j=0}^{n} T_j \varphi(n-j), \qquad n \geq 0,$$

and

$$L(x)=T(x)\phi(x), \tag{4.10}$$

where

$$T(x)=\sum_{n=0}^{\infty} T_n x^n \quad \text{and} \quad \phi(x)=\sum_{n=0}^{\infty} \varphi(n)x^n,$$

for $|x|$ sufficiently small. We shall compute $L(x)$ by determining $T(x)$ and $\phi(x)$.

First, since $S_0=\frac{1}{2}$,

$$\begin{aligned} T(x) &= \sum_{n=0}^{\infty} S_{2n}4^{-n}x^{2n}=\tfrac{1}{2}+\sum_{n=1}^{\infty} 4^{-n}x^{2n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2n}} \\ &= \tfrac{1}{2}+\sum_{k=1}^{\infty}(-1)^{k+1}\sum_{n=1}^{\infty}\left(\frac{x}{2k}\right)^{2n} \\ &= \tfrac{1}{2}+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^2}{4k^2-x^2} \\ &= \frac{\pi x}{4\sin(\pi x/2)}=\frac{x}{4}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{2-x}{2}\right)=\tfrac{1}{2}\Gamma\left(\frac{2+x}{2}\right)\Gamma\left(\frac{2-x}{2}\right). \end{aligned} \tag{4.11}$$

Next, since

$$(1-x)^{-1/2}=\sum_{k=0}^{\infty}\binom{2k}{k}4^{-k}x^k, \qquad -1 \leq x < 1, \tag{4.12}$$

we find that, for $v<1$,

$$\begin{aligned} \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{4^k(2k+1-v)} &= \int_0^1 (1-x^2)^{-1/2}x^{-v}\,dx \\ &= \frac{1}{2}\int_0^1 (1-u)^{-1/2}u^{-(v+1)/2}\,du \\ &= \frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{1-v}{2}\right)}{2\Gamma\left(\frac{2-v}{2}\right)}=\frac{\sqrt{\pi}\,\Gamma\left(\frac{1-v}{2}\right)}{2\Gamma\left(\frac{2-v}{2}\right)}. \end{aligned}$$

Differentiating n times with respect to v, we find that

$$\frac{2}{\pi}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{4^k(2k+1-v)^{n+1}}=\frac{1}{\sqrt{\pi}\,n!}\frac{d^n}{dv^n}\left(\frac{\Gamma\left(\frac{1-v}{2}\right)}{\Gamma\left(\frac{2-v}{2}\right)}\right).$$

Setting $v=0$ yields

$$\varphi(n)=\frac{1}{\sqrt{\pi}\,n!}\frac{d^n}{dv^n}\left(\frac{\Gamma\left(\frac{1-v}{2}\right)}{\Gamma\left(\frac{2-v}{2}\right)}\right)_{v=0}$$

Hence,

$$\phi(x)=\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1-x}{2}\right)}{\Gamma\left(\frac{2-x}{2}\right)}. \tag{4.13}$$

Therefore, by (4.10), (4.11), and (4.13),

$$L(x)=\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{2+x}{2}\right)\Gamma\left(\frac{1-x}{2}\right). \tag{4.14}$$

We next compute $R(x)$. By (4.12),

$$(1+x)^{-1/2}=\sum_{k=0}^{\infty}\binom{2k}{k}(-4)^{-k}x^k, \qquad -1<x\le 1. \tag{4.15}$$

Thus, for $v<1$,

$$\begin{aligned}\frac{1}{2}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{4^k(2k+1-v)}&=\frac{1}{2}\int_0^1(1+x^2)^{-1/2}x^{-v}\,dx\\&=\frac{1}{4}\int_0^1(1+u)^{-1/2}u^{-(v+1)/2}\,du\equiv F(v),\end{aligned} \tag{4.16}$$

say. Differentiating n times with respect to v, we find that

$$\frac{1}{2}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{4^k(2k+1-v)^{n+1}}=\frac{F^{(n)}(v)}{n!}.$$

Recalling that R_n denotes the right side of (4.7), we deduce that

$$R_n=\frac{F^{(n-1)}(0)}{(n-1)!}+\frac{(-1)^nF^{(n-1)}(-1)}{(n-1)!}-\frac{(-1)^nF^{(n)}(-1)}{n!}, \qquad n>0,$$

and, with the help of (4.15),

$$R_0=\frac{1}{\sqrt{2}}-F(-1).$$

Thus,

$$R(x)=\frac{1}{\sqrt{2}}+\sum_{n=1}^{\infty}\left\{\frac{F^{(n-1)}(0)}{(n-1)!}+\frac{(-1)^n F^{(n-1)}(-1)}{(n-1)!}\right\}x^n$$
$$-\sum_{n=0}^{\infty}\frac{F^{(n)}(-1)}{n!}(-x)^n$$
$$=\frac{1}{\sqrt{2}}+xF(x)-xF(-x-1)-F(-x-1)$$
$$=\frac{1}{\sqrt{2}}+xF(x)+(-x-1)F(-x-1). \tag{4.17}$$

To determine $xF(x)$, we return to (4.16), and integrating by parts, we deduce that, for $x<1$,

$$xF(x)=-\frac{1}{2}\int_0^1 (1+u)^{-1/2}u^{1/2}d(u^{-x/2})$$
$$=-\frac{1}{2\sqrt{2}}+\frac{1}{4}\int_0^1 (1+u)^{-3/2}u^{-(x+1)/2}\,du. \tag{4.18}$$

Furthermore, replacing u by $1/u$, we also find that, for $x<1$,

$$xF(x)=-\frac{1}{2\sqrt{2}}+\frac{1}{4}\int_1^{\infty}(1+u)^{-3/2}u^{x/2}\,du. \tag{4.19}$$

Hence, from (4.18) and (4.19), if $-2<x<1$,

$$xF(x)+(-x-1)F(-x-1)=-\frac{1}{\sqrt{2}}+\frac{1}{4}\int_0^{\infty}(1+u)^{-3/2}u^{x/2}\,du$$
$$=-\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{x+2}{2}\right)\Gamma\left(\frac{1-x}{2}\right).$$

We therefore conclude from (4.17) that

$$R(x)=\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{x+2}{2}\right)\Gamma\left(\frac{1-x}{2}\right).$$

By (4.14) and the foregoing equality, we deduce (4.8), which completes the proof.

Proposition 4. *For* $|x|<\pi/2$, *we have*

(i1) $$S_1\equiv\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\sin(2k+1)x=\frac{\sin(x/2)}{\sqrt{2\cos x}},$$

(ii1) $$C_1\equiv\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\cos(2k+1)x=\frac{\cos(x/2)}{\sqrt{2\cos x}},$$

(i2) $$S_2 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k)!}{2^{2k}(k!)^2} \sin(2kx) = \frac{\sin(x/2)}{\sqrt{2\cos x}},$$

(ii2) $$C_2 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \cos(2kx) = \frac{\cos(x/2)}{\sqrt{2\cos x}},$$

(iii) $$S_3 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \sin(2k+2)x = \frac{\sin(3x/2)}{\sqrt{2\cos x}},$$

(iv) $$C_3 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \cos(2k+2)x = \frac{\cos(3x/2)}{\sqrt{2\cos x}},$$

(v) $$S_4 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \frac{\sin(2k+2)x}{2k+2} = \sin\left(\frac{x}{2}\right)\sqrt{2\cos x},$$

(vi) $$C_4 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \frac{\cos(2k+2)x}{2k+2} = \cos\left(\frac{x}{2}\right)\sqrt{2\cos x} - 1,$$

(vii) $$S_5 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \frac{\sin(2k+1)x}{2k+1} = \sin^{-1}\left(\sqrt{2}\sin\left(\frac{x}{2}\right)\right),$$

(viii) $$C_5 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \frac{\cos(2k+1)x}{2k+1} = \operatorname{Log}\left(\sqrt{\cos x} + \sqrt{2}\cos\left(\frac{x}{2}\right)\right),$$

(ix) $$S_6 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \frac{\sin(2k+2)x}{(2k+2)^2}$$
$$= \sin\left(\frac{x}{2}\right)\sqrt{2\cos x} + \sin^{-1}\left(\sqrt{2}\sin\left(\frac{x}{2}\right)\right) - x,$$

and

(x) $$C_6 \equiv \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} \frac{\cos(2k+2)x}{(2k+2)^2}$$
$$= \cos\left(\frac{x}{2}\right)\sqrt{2\cos x} - \operatorname{Log}\left(\sqrt{\cos x} + \sqrt{2}\cos\left(\frac{x}{2}\right)\right) + \operatorname{Log} 2 - 1.$$

In all of the equalities of Proposition 4 and throughout all of their proofs below, we take the principal branches of all multi-valued relations.

Proof of (i1), (ii1). For $|x| < \pi/2$,

$$C_1 + iS_1 = \sum_{k=0}^{\infty} \frac{(-1)^{k}(2k)!}{2^{2k}(k!)^2} e^{(2k+1)ix}$$
$$= e^{ix}(1 + e^{2ix})^{-1/2}$$
$$= \left(\cos\left(\frac{x}{2}\right) + i\sin\left(\frac{x}{2}\right)\right)(2\cos x)^{-1/2}.$$

Equating real and imaginary parts above yields (ii1) and (i1), respectively.

Proof of (i2), (ii2). For $|x| < \pi/2$,

$$\begin{aligned} C_2 - iS_2 &= \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} e^{2kix} \\ &= (1 + e^{2ix})^{-1/2} \\ &= \left(\cos\left(\frac{x}{2}\right) - i \sin\left(\frac{x}{2}\right)\right)(2 \cos x)^{-1/2}, \end{aligned}$$

and the results follow as before.

Proof of (iii), (iv). For $|x| < \pi/2$,

$$C_3 + iS_3 = \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} e^{(2k+2)ix} = e^{2ix}(1 + e^{2ix})^{-1/2},$$

from which the desired equalities readily follow.

Proof of (v), (vi). For $|x| < \pi/2$,

$$\begin{aligned} C_4 + iS_4 &= \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} \frac{e^{(2k+2)ix}}{2k+2} \\ &= (1 + e^{2ix})^{1/2} - 1 = e^{ix/2}\sqrt{2 \cos x} - 1. \end{aligned}$$

Equating real and imaginary parts on both sides above, we complete the proof.

Proof of (vii), (viii). For $|x| < \pi/2$,

$$\begin{aligned} C_5 + iS_5 &= \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} \frac{e^{(2k+1)ix}}{2k+1} \\ &= \sinh^{-1}(e^{ix}) \\ &= \operatorname{Log}(e^{ix} + \sqrt{1 + e^{2ix}}) \\ &= \operatorname{Log}(e^{ix} + e^{ix/2}\sqrt{2 \cos x}). \end{aligned}$$

Hence,

$$\begin{aligned} C_5 &= \tfrac{1}{2} \operatorname{Log}\left\{\left(\cos x + \cos\left(\frac{x}{2}\right)\sqrt{2 \cos x}\right)^2 + \left(\sin x + \sin\left(\frac{x}{2}\right)\sqrt{2 \cos x}\right)^2\right\} \\ &= \tfrac{1}{2} \operatorname{Log}\left\{1 + 2 \cos x + 2 \cos\left(\frac{x}{2}\right)\sqrt{2 \cos x}\right\} \\ &= \tfrac{1}{2} \operatorname{Log}\left\{\cos x + 2 \cos^2\left(\frac{x}{2}\right) + 2 \cos\left(\frac{x}{2}\right)\sqrt{2 \cos x}\right\} \\ &= \operatorname{Log}\left(\sqrt{\cos x} + \sqrt{2} \cos\left(\frac{x}{2}\right)\right), \end{aligned} \tag{4.20}$$

and

$$S_5 = \tan^{-1}\left(\frac{\sin x + \sin(x/2)\sqrt{2\cos x}}{\cos x + \cos(x/2)\sqrt{2\cos x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\sin(x/2)\{\sqrt{2}\cos(x/2) + \sqrt{\cos x}\}}{\sqrt{\cos x}\{\sqrt{\cos x} + \sqrt{2}\cos(x/2)\}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\sin(x/2)}{\sqrt{1 - 2\sin^2(x/2)}}\right)$$

$$= \sin^{-1}\left(\sqrt{2}\sin\left(\frac{x}{2}\right)\right). \tag{4.21}$$

Proof of (ix), (x). For $|x| < \pi/2$, put

$$C_6 + iS_6 = \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} \frac{e^{(2k+2)ix}}{(2k+2)^2} = f(u),$$

where $u = e^{ix}$. Observe that

$$uf'(u) = \sqrt{1 + u^2} - 1.$$

Thus,

$$f(t) = \int_0^t \frac{1}{u}(\sqrt{1 + u^2} - 1)\, du$$

$$= \{\sqrt{1 + u^2} - \operatorname{Log}(1 + \sqrt{1 + u^2})\}\Big|_0^t$$

$$= \sqrt{1 + t^2} - 1 + \operatorname{Log} 2 + \operatorname{Log} t - \operatorname{Log}\left(\frac{1}{t} + \sqrt{t^{-2} + 1}\right).$$

Hence,

$$C_6 + iS_6 = e^{ix/2}\sqrt{2\cos x} - 1 + \operatorname{Log} 2 - ix$$
$$- \tfrac{1}{2}\operatorname{Log}\left\{\left(\cos x + \cos\left(\frac{x}{2}\right)\sqrt{2\cos x}\right)^2\right.$$
$$\left. + \left(\sin x + \sin\left(\frac{x}{2}\right)\sqrt{2\cos x}\right)^2\right\}$$
$$+ i\tan^{-1}\left(\frac{\sin x + \sin(x/2)\sqrt{2\cos x}}{\cos x + \cos(x/2)\sqrt{2\cos x}}\right)$$
$$= e^{ix/2}\sqrt{2\cos x} - 1 + \operatorname{Log} 2 - ix$$
$$- \operatorname{Log}\left(\sqrt{\cos x} + \sqrt{2}\cos\left(\frac{x}{2}\right)\right) + i\sin^{-1}\left(\sqrt{2}\sin\left(\frac{x}{2}\right)\right),$$

by (4.20) and (4.21). Equating real and imaginary parts above, we deduce (x) and (ix), respectively.

Entry 5. *Let a, n, and θ be real with $n \geq 0$ and $|\theta| \leq \pi/2$. Then*

(i) $$S \equiv \sum_{k=0}^{\infty} \binom{n}{k} \sin(a+2k)\theta = 2^n \cos^n \theta \sin(a+n)\theta$$

and

(ii) $$C \equiv \sum_{k=0}^{\infty} \binom{n}{k} \cos(a+2k)\theta = 2^n \cos^n \theta \cos(a+n)\theta.$$

Proof. By Stirling's formula (I6), the series in (i) and (ii), indeed, do converge (absolutely) for $n \geq 0$. Now,

$$C + iS = \sum_{k=0}^{\infty} \binom{n}{k} e^{(a+2k)i\theta} = e^{ia\theta}(1+e^{2i\theta})^n$$
$$= e^{i(a+n)\theta}(2\cos\theta)^n.$$

Equating real and imaginary parts on both sides above, we deduce (ii) and (i), respectively.

If $|z| \leq 1$ and n is a natural number with $n \geq 2$, the polylogarithm $\mathrm{Li}_n(z)$ is defined by

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \tag{6.1}$$

Furthermore, set

$$2\chi_n(z) = \mathrm{Li}_n(z) - \mathrm{Li}_n(-z) = 2\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^n}. \tag{6.2}$$

Observe that, for $|z| \leq 1$,

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{Log}(1-w)}{w}\,dw = \int_0^z \frac{dw}{w}\int_0^w \frac{du}{1-u}, \tag{6.3}$$

where the principal branch of $\mathrm{Log}(1-w)$ is assumed. (The latter expression for $\mathrm{Li}_2(z)$ suggests the terminology "dilogarithm.") Equality (6.3) may be used to define $\mathrm{Li}_2(z)$ for all complex z. By employing the equality

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(w)\,dw}{w},$$

we may, by induction, analytically continue $\mathrm{Li}_n(z)$, $n \geq 2$, to the entire complex z-plane, cut along $[1, \infty)$. In the sequel, we shall often say that certain functions related to the dilogarithm have analytic continuations to the entire complex plane. It is to be tacitly understood that the analytically continued functions will generally be holomorphic except on a branch cut. In Sections 6 and 7, Ramanujan derives several properties of the dilogarithm $\mathrm{Li}_2(z)$ and trilogarithm $\mathrm{Li}_3(z)$. Since most of these results are known, we shall not give complete proofs but refer to Lewin's book [1] where proofs may be found.

Recall that the Bernoulli numbers B_n, $0 \leq n < \infty$, are defined by (I1).

Entry 6. *Let* Li_2 *and* χ_2 *be defined as above. Then*

(i) $$\mathrm{Li}_2(1-z) + \mathrm{Li}_2\left(1 - \frac{1}{z}\right) = -\tfrac{1}{2}\,\mathrm{Log}^2 z,$$

(ii) $$\mathrm{Li}_2(-z) + \mathrm{Li}_2\left(\frac{-1}{z}\right) = -\frac{\pi^2}{6} - \tfrac{1}{2}\,\mathrm{Log}^2 z,$$

(iii) $$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \mathrm{Log}\, z\, \mathrm{Log}(1-z),$$

(iv) $$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \tfrac{1}{2}\,\mathrm{Li}_2(z^2),$$

(v) $$\chi_2(z) + \chi_2\left(\frac{1-z}{1+z}\right) = \frac{\pi^2}{8} + \tfrac{1}{2}\,\mathrm{Log}\, z\, \mathrm{Log}\left(\frac{1+z}{1-z}\right),$$

(vi) $$\mathrm{Li}_2\left(\frac{z}{1-w}\right) + \mathrm{Li}_2\left(\frac{w}{1-z}\right) = \mathrm{Li}_2(z) + \mathrm{Li}_2(w) + \mathrm{Li}_2\left(\frac{zw}{(1-z)(1-w)}\right) + \mathrm{Log}(1-z)\,\mathrm{Log}(1-w),$$

(vii) $$\mathrm{Li}_2(e^{-z}) = \frac{\pi^2}{6} + z\,\mathrm{Log}\, z - z + \sum_{n=1}^{\infty} \frac{B_n z^{n+1}}{(n+1)!n}, \qquad |z| < 2\pi,$$

and

(viii) $$\mathrm{Li}_2(1 - e^{-z}) = \sum_{n=0}^{\infty} \frac{B_n z^{n+1}}{(n+1)!}, \qquad |z| < 2\pi.$$

Proof. Part (i) is proved in Lewin's text [1, p. 5, equation (1.12)] and is due to Landen [1].

Part (ii) is also found in Lewin's book [1, p. 4, equation (1.7)] on noting that $\mathrm{Li}_2(-1) = -\pi^2/12$. Evidently, (ii) is due to Euler [5, p. 38], [9, p. 133]. See the top of page 5 of Lewin's text [1] for further references.

Equality (iii) is also due to Euler [5], [9, p. 130] and can be found in Lewin's book [1, p. 5, equation (1.11)].

Formula (iv) is rather trivial and can be found in Lewin's book [1, p. 6, equation (1.15)].

Part (v) is again due to Landen [1] and is established in Lewin's treatise [1, p. 19, equation (1.67)].

Formula (vi) was first established by Abel [3, p. 193], but an equivalent formula was proved earlier by Spence [1]. The former formula is also in Lewin's book [1, p. 8, equation (1.22)].

Formula (vii) arises from (6.4) after dividing both sides of (6.4) by z and integrating both sides twice. See Lewin's book [1, p. 21, equation (1.76)].

To prove (viii), first replace z by t in (6.4) and integrate both sides over $[0, z]$. Next, in the resulting integral on the left side, set $w = 1 - e^{-t}$. Using (6.3), we complete the proof.

Example. *We have*

(i)
$$\mathrm{Li}_2(\tfrac{1}{2}) = \frac{\pi^2}{12} - \tfrac{1}{2}\,\mathrm{Log}^2\, 2,$$

(ii)
$$\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \mathrm{Log}^2\left(\frac{\sqrt{5}-1}{2}\right),$$

(iii)
$$\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} - \mathrm{Log}^2\left(\frac{\sqrt{5}-1}{2}\right),$$

(iv)
$$\chi_2(\sqrt{2}-1) = \frac{\pi^2}{16} - \tfrac{1}{4}\,\mathrm{Log}^2(\sqrt{2}-1),$$

(v)
$$\chi_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{12} - \tfrac{3}{4}\,\mathrm{Log}^2\left(\frac{\sqrt{5}-1}{2}\right),$$

and

(vi)
$$\chi_2(\sqrt{5}-2) = \frac{\pi^2}{24} - \tfrac{3}{4}\,\mathrm{Log}^2\left(\frac{\sqrt{5}-1}{2}\right).$$

Proof. Part (i) follows from Entry 6(iii) on setting $z = \frac{1}{2}$. The result is found in Lewin's book [1, p. 6, equation (1.16)]. The priority for this evaluation seems to be clouded. According to Lewin [1, p. 6], the result is credited to Euler in 1761, but Landen claims to have established (i) in 1760. On the other hand, Bromwich [1, p. 520] indicates that the result is due to Legendre.

Formula (ii) can be found in Lewin's treatise [1, p. 7, equation (1.20)] and is apparently due to Landen [1].

Formula (iii) is found in Lewin's book [1, p. 7] and is again due to Landen [1].

Equality (iv), which readily follows from Entry 6(v) upon setting $z = \sqrt{2} - 1$, is again found in Lewin's book [1, p. 19, equation (1.68)].

Part (v) is also found in Lewin's book [1, p. 19, equation (1.69)] and is due to Landen [1].

Formula (vi) was submitted by Ramanujan as a problem in the *Journal of the Indian Mathematical Society* [8], [15, p. 330]. See also Lewin's book [1, p. 19, equation (1.70)], where the result is attributed to Landen [1].

Many other functional equations and numerical examples for the dilogarithm can be found in papers of Euler [2], [5], Gastmans and Troost [1], Lewin [2], Loxton [1], Richmond and Szekeres [1], and Schaeffer [1] and the books of Landen [1], Lewin [1], and Spence [1].

Entry 7. *Let* $\mathrm{Li}_3(z)$ *be defined by* (6.1). *Then*

(i)
$$\mathrm{Li}_3(1-z)+\mathrm{Li}_3\left(1-\frac{1}{z}\right)+\mathrm{Li}_3(z)$$
$$=\zeta(3)+\frac{\pi^2}{6}\operatorname{Log} z+\tfrac{1}{6}\operatorname{Log}^3 z-\tfrac{1}{2}\operatorname{Log}^2 z\operatorname{Log}(1-z),$$

(ii)
$$\mathrm{Li}_3(-z)-\mathrm{Li}_3\left(\frac{-1}{z}\right)=-\tfrac{1}{6}\operatorname{Log}^3 z-\frac{\pi^2}{6}\operatorname{Log} z,$$

and

(iii)
$$\mathrm{Li}_3(z)+\mathrm{Li}_3(-z)=\tfrac{1}{4}\mathrm{Li}_3(z^2).$$

Proof. Part (i) is due to Landen [1] and can be deduced from Lewin's book [1, p. 155, equation (6.10)] by letting $x=1-z$ there.

Part (ii) can be found in Lewin's book [1, p. 154, equation (6.6)].

Part (iii) is trivial. See also Lewin's treatise [1, p. 154, equation (6.4)].

Example. *If* χ_3 *is defined by* (6.2), *then*

(i)
$$\mathrm{Li}_3(\tfrac{1}{2})=\tfrac{1}{6}\operatorname{Log}^3 2-\frac{\pi^2}{12}\operatorname{Log} 2+\chi_3(1)$$

and

(ii)
$$\mathrm{Li}_3\left(\frac{3-\sqrt{5}}{2}\right)=\tfrac{2}{3}\operatorname{Log}^3\left(\frac{\sqrt{5}+1}{2}\right)-\frac{2\pi^2}{15}\operatorname{Log}\left(\frac{\sqrt{5}+1}{2}\right)+\tfrac{4}{5}\zeta(3).$$

Proof. The first equality follows from Entry 7(i) on setting $z=\frac{1}{2}$. See also Lewin's text [1, p. 155, equation (6.12)].

Part (ii), due to Landen [1], is again in Lewin's book [1, p. 156, equation (6.13)]. In Ramanujan's notebooks, p. 107, the coefficient $\frac{4}{5}$ on the right side of (ii) is inadvertently omitted.

Entry 8. *For* $|x|<1$, *define*

$$f(x)=\sum_{k=1}^{\infty}\frac{h_k x^{2k-1}}{2k-1}, \tag{8.1}$$

where

$$h_n=\sum_{k=1}^{n}\frac{1}{2k-1}. \tag{8.2}$$

Then, for $|x| < 1$,

$$f\left(\frac{x}{2-x}\right) = \tfrac{1}{8}\operatorname{Log}^2(1-x) + \tfrac{1}{2}\operatorname{Li}_2(x).$$

Proof. Taking the Cauchy product of the Maclaurin series for $1/(1-x^2)$ and $\operatorname{Log}\{(1+x)/(1-x)\}$, we find that, for $|x| < 1$,

$$f'(x) = \frac{1}{1-x^2}\frac{1}{2x}\operatorname{Log}\left(\frac{1+x}{1-x}\right).$$

Hence,

$$\begin{aligned}\frac{2}{(2-x)^2} f'\left(\frac{x}{2-x}\right) &= \frac{x-2}{4x(1-x)}\operatorname{Log}(1-x)\\ &= -\frac{1}{2x}\operatorname{Log}(1-x) - \frac{\operatorname{Log}(1-x)}{4(1-x)}.\end{aligned}$$

Integrating the foregoing equality over $[0, x]$, $|x| < 1$, and using (6.3) and the equality

$$\frac{d}{dx}\left(\frac{x}{2-x}\right) = \frac{2}{(2-x)^2},$$

we complete the proof.

Example. *With* f *defined by* (8.1), *we have*

(i) $$f(\tfrac{1}{3}) = \frac{\pi^2}{24} - \tfrac{1}{8}\operatorname{Log}^2 2,$$

(ii) $$f\left(\frac{1}{\sqrt{5}}\right) = \frac{\pi^2}{20},$$

and

(iii) $$f(\sqrt{5}-2) = \frac{\pi^2}{30} - \tfrac{3}{8}\operatorname{Log}^2\left(\frac{\sqrt{5}-1}{2}\right).$$

Proof. To deduce part (i), set $x = \frac{1}{2}$ in Entry 8 and then employ Example (i) of Section 6.

To obtain (ii), set $x = (\sqrt{5}-1)/2$ in Entry 8, note that $1 - x = x^2$, and use Example (ii) of Section 6.

Lastly, set $x = (3-\sqrt{5})/2$ in Entry 8. Using Example (iii) of Section 6, we deduce the desired equality.

Both Examples (ii) and (iii) are in error in the notebooks, p. 107. In (ii), Ramanujan has an extra term $-\frac{3}{8}\operatorname{Log}^2\{(\sqrt{5}-1)/2\}$ on the right side. In (iii), he has written $\frac{3}{4}$ instead of $\frac{3}{8}$ on the right side. Ramanujan [9], [15, p. 330] submitted Examples (i) and (ii) as a question in the *Journal of the Indian Mathematical Society.*

For other examples of this sort, see Catalan's paper [1].

Entry 9. *For* $|z| \le 1$, *define*

$$g(z) = \sum_{k=1}^{\infty} \frac{H_k z^{k+1}}{(k+1)^2}, \tag{9.1}$$

where H_k *is defined by* (3.1). *Then* g *can be analytically continued to the entire complex plane. Furthermore,*

(i)
$$g(1-z) = \tfrac{1}{2}\operatorname{Log}^2 z \operatorname{Log}(1-z) + \operatorname{Li}_2(z)\operatorname{Log} z - \operatorname{Li}_3(z) + \zeta(3),$$

(ii)
$$g(1-z) - g(1-1/z) = \tfrac{1}{6}\operatorname{Log}^3 z,$$

(iii)
$$g(1-z) = \tfrac{1}{2}\operatorname{Log}^2 z \operatorname{Log}(z-1) - \tfrac{1}{3}\operatorname{Log}^3 z - \operatorname{Li}_2\left(\frac{1}{z}\right)\operatorname{Log} z - \operatorname{Li}_3\left(\frac{1}{z}\right) + \zeta(3),$$

and

(iv)
$$g(-z) + g\left(\frac{-1}{z}\right) = -\tfrac{1}{6}\operatorname{Log}^3 z - \operatorname{Li}_2(-z)\operatorname{Log} z + \operatorname{Li}_3(-z) + \zeta(3).$$

Proof. Squaring the Maclaurin series for $\operatorname{Log}(1-z)$, we find that

$$zg'(z) = \tfrac{1}{2}\operatorname{Log}^2(1-z), \qquad |z| < 1. \tag{9.2}$$

Thus,

$$-g'(1-z) = -\frac{\operatorname{Log}^2 z}{2(1-z)}, \qquad 0 < z < 1. \tag{9.3}$$

Integrating by parts twice, we find that

$$g(1-z) = \tfrac{1}{2}\operatorname{Log}^2 z \operatorname{Log}(1-z) + \operatorname{Li}_2(z)\operatorname{Log} z - \operatorname{Li}_3(z) + c.$$

If we let z tend to 1^-, we find that $c = \operatorname{Li}_3(1) = \zeta(3)$, which completes the proof of (i) for $0 < z < 1$.

Since $\operatorname{Li}_2(z)$ and $\operatorname{Li}_3(z)$ can be analytically continued into the full complex z-plane, then (i) shows that $g(z)$ can be analytically continued as well.

From (9.3),

$$-g'(1-z) - \frac{1}{z^2}g'\left(1-\frac{1}{z}\right) = \frac{\operatorname{Log}^2 z}{2z}.$$

Integrating this equality over $[1, z]$, we get (ii).

Consider (i) for $0 < z < 1$. Replacing z by $1/z$, we obtain for $z > 1$,

$$g\left(1 - \frac{1}{z}\right) = \tfrac{1}{2}\operatorname{Log}^2 z\{\operatorname{Log}(z-1) - \operatorname{Log} z\} - \operatorname{Li}_2\left(\frac{1}{z}\right)\operatorname{Log} z - \operatorname{Li}_3\left(\frac{1}{z}\right) + \zeta(3).$$

Substituting this expression for $g(1 - 1/z)$ into (ii), we deduce (iii) for $z > 1$. By analytic continuation, (iii) is valid for all complex z.

Lastly, if $z > 0$, we find from (9.2) that

$$-g'(-z) + \frac{1}{z^2}g'\left(\frac{-1}{z}\right) = \frac{\operatorname{Log}^2(1+z)}{2z} - \frac{\operatorname{Log}^2(1+1/z)}{2z} = -\frac{\operatorname{Log}^2 z}{2z} + \frac{\operatorname{Log} z \operatorname{Log}(1+z)}{z}.$$

Integrating by parts, we find that

$$g(-z) + g\left(\frac{-1}{z}\right) = -\frac{\operatorname{Log}^3 z}{6} - \operatorname{Li}_2(-z)\operatorname{Log} z + \int \frac{\operatorname{Li}_2(-z)}{z}\,dz = -\frac{\operatorname{Log}^3 z}{6} - \operatorname{Li}_2(-z)\operatorname{Log} z + \operatorname{Li}_3(-z) + c.$$

By analytic continuation, this holds for all complex z. Now set $z = -1$ and use the fact that $g(1) = \zeta(3)$, which can be deduced from (i). We then find that $c = \zeta(3)$, and so the proof of (iv) is complete.

Entry 9(i) is stated without proof by Lewin [1, p. 303, formula (12)]. Entry 9(iii) contains a misprint in the notebooks, p. 107; Ramanujan has written $\operatorname{Log}(1-z)$ for $\operatorname{Log}(z-1)$ on the right side.

The formula

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} = \zeta(3), \tag{9.4}$$

obtained from Entry 9(i) by setting $z = 0$, has a long history. Formula (9.4) was evidently first discovered by Euler [2], [8, p. 228] in 1775. This evaluation and many other results of this sort were established by Nielsen [3], [4], [6]. In 1952, (9.4) was rediscovered by Klamkin [1] and submitted as a problem. Briggs, Chowla, Kempner, and Mientka [1] rediscovered the result again in 1955. Once again, in 1982, (9.4) was rediscovered, by Bruckman [1]. In fact, (9.4) is a special case of the more general formula

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^n} = (n+2)\zeta(n+1) - \sum_{k=1}^{n-2} \zeta(n-k)\zeta(k+1), \tag{9.5}$$

where n is a positive integer at least equal to 2. This result is also due to Euler

[2], [8, p. 266]. Nielsen [3, p. 229], [4, p. 198], [6, pp. 47–49] developed a very general method for obtaining (9.5) and similar types of results. Formula (9.5) was rediscovered by Williams [2] in 1953. Sitaramachandrarao and Siva Rama Sarma [1] and Georghiou and Philippou [1] have also proved (9.5).

It might be remarked that the problem of evaluating series of the type

$$\sum_{k=1}^{\infty} \frac{1}{k^n} \sum_{j=1}^{k} \frac{1}{j^m},$$

where m and n are positive integers with $n \geq 2$, was first proposed in a letter from Goldbach to Euler [10] in 1742. (See also Euler and Goldbach [1].) The two mathematicians exchanged a series of letters about this problem in 1742 and 1743, and Euler was successful in obtaining several evaluations of series like that depicted above. However, (9.4) and (9.5) apparently are not found in these letters.

Matsuoka [1] and Apostol and Vu [1] have made a study of the related Dirichlet series

$$F(s) = \sum_{k=1}^{\infty} H_{k-1} k^{-s}, \qquad \operatorname{Re}(s) > 1.$$

In particular, they have shown that F can be analytically continued to the entire complex s-plane. Apostol and Vu have further studied more general as well as other related series.

Entry 10. *For* $|z| \leq 1$, *define*

$$h(z) = \sum_{k=1}^{\infty} \frac{H_k z^{k+1}}{(k+1)^3}.$$

Then $h(z)$ *can be analytically continued into the entire complex z-plane. Furthermore,*

$$\text{(i)} \quad h(1-z) - h\left(1 - \frac{1}{z}\right) = -\tfrac{1}{24} \operatorname{Log}^4 z + \tfrac{1}{6} \operatorname{Log}^3 z \operatorname{Log}(1-z) + \zeta(3) \operatorname{Log} z - 2 \operatorname{Li}_4(z) + \operatorname{Li}_3(z) \operatorname{Log} z + \frac{\pi^2}{45}$$

and

$$\text{(ii)} \quad h(-z) - h\left(\frac{-1}{z}\right) = -\tfrac{1}{24} \operatorname{Log}^4 z - \operatorname{Li}_3(-z) \operatorname{Log} z + 2 \operatorname{Li}_4(-z) + \zeta(3) \operatorname{Log} z + \frac{7\pi^4}{360}.$$

Proof. First observe that

$$h'(z) = \frac{g(z)}{z}, \tag{10.1}$$

where g is defined by (9.1). It follows from Entry 9 that h can be analytically continued into the entire complex z-plane.

By (10.1), Entry 9(ii), and Entry 9(i),

$$\begin{aligned}
-h'(1-z)-\frac{1}{z^2}h'\left(1-\frac{1}{z}\right) &= -\frac{g(1-z)}{1-z}-\frac{g(1-1/z)}{z^2(1-1/z)} \\
&= -\frac{1}{1-z}\left\{g(1-z)-\frac{1}{z}g(1-z)+\frac{1}{6z}\operatorname{Log}^3 z\right\} \\
&= \frac{g(1-z)}{z}-\frac{\operatorname{Log}^3 z}{6z(1-z)} \\
&= \frac{1}{2z}\operatorname{Log}^2 z \operatorname{Log}(1-z) \\
&\quad +\frac{1}{z}\operatorname{Li}_2(z)\operatorname{Log} z-\frac{1}{z}\operatorname{Li}_3(z) \\
&\quad +\frac{1}{z}\zeta(3)-\frac{1}{6z}\operatorname{Log}^3 z-\frac{1}{6(1-z)}\operatorname{Log}^3 z.
\end{aligned}$$

Integrating the equality above, we find that

$$\begin{aligned}
h(1-z)-h\left(1-\frac{1}{z}\right) &= \int\frac{\operatorname{Log}^2 z\operatorname{Log}(1-z)}{2z}\,dz+\operatorname{Li}_3(z)\operatorname{Log} z \\
&\quad -\int\frac{\operatorname{Li}_3(z)}{z}\,dz-\int\frac{\operatorname{Li}_3(z)}{z}\,dz+\zeta(3)\operatorname{Log} z-\tfrac{1}{24}\operatorname{Log}^4 z \\
&\quad +\tfrac{1}{6}\operatorname{Log}^3 z\operatorname{Log}(1-z)-\int\frac{\operatorname{Log}^2 z\operatorname{Log}(1-z)}{2z}\,dz+c \\
&= \operatorname{Li}_3(z)\operatorname{Log} z-2\operatorname{Li}_4(z)+\zeta(3)\operatorname{Log} z-\tfrac{1}{24}\operatorname{Log}^4 z \\
&\quad +\tfrac{1}{6}\operatorname{Log}^3 z\operatorname{Log}(1-z)+c,
\end{aligned}$$

where in the penultimate equality we integrated by parts twice. Letting $z=1$ and employing the fact that $\operatorname{Li}_4(1)=\zeta(4)=\pi^4/90$, we find that $c=\pi^4/45$, which completes the proof of (i).

Next, by (10.1),

$$\begin{aligned}
-h'(-z)-\frac{1}{z^2}h'\left(\frac{-1}{z}\right) &= \frac{1}{z}\left\{g(-z)+g\left(\frac{-1}{z}\right)\right\} \\
&= -\frac{1}{6z}\operatorname{Log}^3 z-\frac{1}{z}\operatorname{Li}_2(-z)\operatorname{Log} z \\
&\quad +\frac{1}{z}\operatorname{Li}_3(-z)+\frac{1}{z}\zeta(3),
\end{aligned}$$

by Entry 9(iv). Integrating both sides above, we find that

$$h(-z)-h\left(\frac{-1}{z}\right) = -\tfrac{1}{24}\operatorname{Log}^4 z - \operatorname{Li}_3(-z)\operatorname{Log} z + \int \frac{\operatorname{Li}_3(-z)}{z}\,dz + \int \frac{\operatorname{Li}_3(-z)}{z}\,dz + \zeta(3)\operatorname{Log} z + c$$

$$= -\tfrac{1}{24}\operatorname{Log}^4 z - \operatorname{Li}_3(-z)\operatorname{Log} z + 2\operatorname{Li}_4(-z) + \zeta(3)\operatorname{Log} z + c.$$

Putting $z = 1$ above and using the fact that $\operatorname{Li}_4(-1) = -7\zeta(4)/8 = -7\pi^4/720$, we find that $c = 7\pi^4/360$. This completes the proof of (ii).

In the notebooks, pp. 107, 108, the right sides of Entries 10(i) and (ii) must be multiplied by (-1).

Entry 11. *For* $-1 \le x \le 1$, *define*

$$F(x) = \sum_{k=1}^{\infty} \frac{h_k x^{2k}}{(2k)^2} \quad \text{and} \quad G(x) = \sum_{k=1}^{\infty} \frac{h_k x^{2k}}{(2k)^3},$$

where h_k *is defined by* (8.2). *Then for* $0 \le x \le 1$,

(i) $$F\left(\frac{1-x}{1+x}\right) = \tfrac{1}{8}\operatorname{Log}^2 x \operatorname{Log}\left(\frac{1-x}{1+x}\right) + \tfrac{1}{2}\chi_2(x)\operatorname{Log} x + \tfrac{1}{2}\{\chi_3(1) - \chi_3(x)\}$$

and

(ii) $$G(x) + G\left(\frac{1-x}{1+x}\right) = F(x)\operatorname{Log} x + F\left(\frac{1-x}{1+x}\right)\operatorname{Log}\left(\frac{1-x}{1+x}\right) - \tfrac{1}{16}\operatorname{Log}^2 x \operatorname{Log}^2\left(\frac{1-x}{1+x}\right) + G(1).$$

Proof. For $|x| \le 1$,

$$xF'(x) = \sum_{k=1}^{\infty} \frac{h_k x^{2k}}{2k} = \int_0^x \sum_{k=1}^{\infty} h_k t^{2k-1}\,dt$$

$$= \frac{1}{2}\int_0^x \frac{1}{1-t^2}\operatorname{Log}\left(\frac{1+t}{1-t}\right)dt$$

$$= \frac{1}{4}\int_0^x \operatorname{Log}\left(\frac{1+t}{1-t}\right) d\operatorname{Log}\left(\frac{1+t}{1-t}\right)$$

$$= \tfrac{1}{8}\operatorname{Log}^2\left(\frac{1+x}{1-x}\right). \qquad (11.1)$$

Hence, for $0 \le x < 1$,

$$-\frac{2}{(1+x)^2}F'\left(\frac{1-x}{1+x}\right) = -\frac{\operatorname{Log}^2 x}{4(1-x^2)}. \tag{11.2}$$

Integrating the equality above, we find that

$$\begin{aligned}
F\left(\frac{1-x}{1+x}\right) &= \tfrac{1}{8}\operatorname{Log}^2 x \operatorname{Log}\left(\frac{1-x}{1+x}\right) - \int \frac{\operatorname{Log} x}{4x}\operatorname{Log}\left(\frac{1-x}{1+x}\right) dx \\
&= \tfrac{1}{8}\operatorname{Log}^2 x \operatorname{Log}\left(\frac{1-x}{1+x}\right) + \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2k+1}\int x^{2k}\operatorname{Log} x\, dx \\
&= \tfrac{1}{8}\operatorname{Log}^2 x \operatorname{Log}\left(\frac{1-x}{1+x}\right) + \tfrac{1}{2}\chi_2(x)\operatorname{Log} x - \frac{1}{2}\int \frac{\chi_2(x)}{x} dx \\
&= \tfrac{1}{8}\operatorname{Log}^2 x \operatorname{Log}\left(\frac{1-x}{1+x}\right) + \tfrac{1}{2}\chi_2(x)\operatorname{Log} x - \tfrac{1}{2}\chi_3(x) + c.
\end{aligned}$$

Now let x tend to $1-$ to find that $c = \chi_3(1)/2$, which completes the proof of (i).

We now prove (ii). Observe that for $|x| \le 1$, $xG'(x) = F(x)$. Hence, for $0 \le x < 1$,

$$G'(x) - \frac{2}{(1+x)^2}G'\left(\frac{1-x}{1+x}\right) = \frac{F(x)}{x} - \frac{2}{1-x^2}F\left(\frac{1-x}{1+x}\right).$$

An integration of this equality yields

$$\begin{aligned}
G(x) + G\left(\frac{1-x}{1+x}\right) &= F(x)\operatorname{Log} x - \int F'(x)\operatorname{Log} x\, dx \\
&\quad + F\left(\frac{1-x}{1+x}\right)\operatorname{Log}\left(\frac{1-x}{1+x}\right) \\
&\quad + \int \frac{2}{(1+x)^2}F'\left(\frac{1-x}{1+x}\right)\operatorname{Log}\left(\frac{1-x}{1+x}\right) dx \\
&= F(x)\operatorname{Log} x + F\left(\frac{1-x}{1+x}\right)\operatorname{Log}\left(\frac{1-x}{1+x}\right) \\
&\quad - \frac{1}{8}\int \operatorname{Log}^2\left(\frac{1-x}{1+x}\right)\frac{\operatorname{Log} x}{x} dx \\
&\quad + \frac{1}{4}\int \frac{\operatorname{Log}^2 x}{1-x^2}\operatorname{Log}\left(\frac{1-x}{1+x}\right) dx,
\end{aligned}$$

by (11.1) and (11.2). Integrating by parts, we get

$$\begin{aligned}G(x)+G\left(\frac{1-x}{1+x}\right) &= F(x)\operatorname{Log}x+F\left(\frac{1-x}{1+x}\right)\operatorname{Log}\left(\frac{1-x}{1+x}\right)\\ &\quad -\tfrac{1}{16}\operatorname{Log}^2\left(\frac{1-x}{1+x}\right)\operatorname{Log}^2 x\\ &\quad -\frac{1}{4}\int \operatorname{Log}\left(\frac{1-x}{1+x}\right)\frac{\operatorname{Log}^2 x}{1-x^2}\,dx\\ &\quad +\frac{1}{4}\int \frac{\operatorname{Log}^2 x}{1-x^2}\operatorname{Log}\left(\frac{1-x}{1+x}\right)dx\\ &= F(x)\operatorname{Log}x+F\left(\frac{1-x}{1+x}\right)\operatorname{Log}\left(\frac{1-x}{1+x}\right)\\ &\quad -\tfrac{1}{16}\operatorname{Log}^2 x\operatorname{Log}^2\left(\frac{1-x}{1+x}\right)+c.\end{aligned}$$

Letting x approach $1-$, we find that $c=G(1)$, and this completes the proof.

In fact, Ramanujan claims that

$$G(1)=\frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^3}-\frac{\pi}{3\sqrt{3}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}. \tag{11.3}$$

Unfortunately, this beautiful formula is incorrect. Taking the first three terms of the series defining $G(1)$, we find that $G(1)>0.1529320988\ldots$. On the other hand, the right side of (11.3) is easily found to be less than $0.1442780636\ldots$. We have been unable to find any formula for $G(1)$ which resembles (11.3). R. Sitaramachandrarao (personal communication) has derived several expressions for $G(1)$ that are related to the Riemann zeta-function and similar types of series. Unfortunately, none of Sitaramachandrarao's formulas echoes (11.3).

Entry 12. *For* $|x|<1$, *define*

$$H(x)=\sum_{k=1}^{\infty}\frac{H_k x^{2k-1}}{2k-1},$$

where H_k *is defined by* (3.1). *Then for* $0<x<1$,

$$\begin{aligned}H\left(\frac{1-x}{1+x}\right) &= (\operatorname{Log}2-1)\operatorname{Log}x+\frac{1+x}{1-x}\operatorname{Log}\left(\frac{4x}{(1+x)^2}\right)\\ &\quad +\tfrac{1}{4}\operatorname{Log}^2 x+\frac{\pi^2}{12}+\operatorname{Li}_2(-x).\end{aligned}$$

Proof. First observe that, for $|x|<1$, $x^2H'(x)$ is the Cauchy product of the

Maclaurin series for $-\operatorname{Log}(1-x^2)$ and $1/(1-x^2)$, i.e.,

$$H'(x) = -\frac{\operatorname{Log}(1-x^2)}{x^2(1-x^2)}.$$

Hence,

$$-\frac{2}{(1+x)^2} H'\left(\frac{1-x}{1+x}\right) = \frac{(1+x)^2}{2x(1-x)^2} \operatorname{Log}\left(\frac{4x}{(1+x)^2}\right)$$
$$= \left(\frac{1}{2x} + \frac{2}{(1-x)^2}\right)\{\operatorname{Log} 4 + \operatorname{Log} x - 2 \operatorname{Log}(1+x)\}.$$

Integrating the equality above, we find that, for $0 < x < 1$,

$$H\left(\frac{1-x}{1+x}\right) = \tfrac{1}{2} \operatorname{Log} 4 \operatorname{Log} x + \frac{2}{1-x} \operatorname{Log} 4 + \tfrac{1}{4} \operatorname{Log}^2 x + 2 \int \frac{\operatorname{Log} x}{(1-x)^2}\, dx$$
$$- \int \frac{\operatorname{Log}(1+x)}{x}\, dx - 4 \int \frac{\operatorname{Log}(1+x)}{(1-x)^2}\, dx$$
$$= \operatorname{Log} 2 \operatorname{Log} x + \frac{2}{1-x} \operatorname{Log} 4 + \tfrac{1}{4} \operatorname{Log}^2 x$$
$$+ \frac{2 \operatorname{Log} x}{1-x} - 2 \int \frac{dx}{x(1-x)}$$
$$+ \operatorname{Li}_2(-x) - \frac{4 \operatorname{Log}(1+x)}{1-x} + 4 \int \frac{dx}{1-x^2}$$
$$= \operatorname{Log} 2 \operatorname{Log} x + \frac{2 \operatorname{Log}(4x)}{1-x} + \tfrac{1}{4} \operatorname{Log}^2 x - 2 \operatorname{Log} x$$
$$+ 2 \operatorname{Log}(1-x) + \operatorname{Li}_2(-x) - \frac{2 \operatorname{Log}(1+x)^2}{1-x}$$
$$+ 2 \operatorname{Log}(1+x) - 2 \operatorname{Log}(1-x) + c_1$$
$$= (\operatorname{Log} 2 - 1) \operatorname{Log} x + \frac{2}{1-x} \operatorname{Log}\left(\frac{4x}{(1+x)^2}\right) + \tfrac{1}{4} \operatorname{Log}^2 x$$
$$+ \operatorname{Li}_2(-x) - \operatorname{Log}\left(\frac{4x}{(1+x)^2}\right) + c,$$

where $c = c_1 + \operatorname{Log} 4$. Letting x tend to $1-$, we find that $c = -\operatorname{Li}_2(-1) = \pi^2/12$, and this completes the proof.

Examples. *We have*

(i) $$\sum_{k=1}^{\infty} \frac{H_k}{k^2 2^k} = \zeta(3) - \frac{\pi^2}{12} \operatorname{Log} 2,$$

(ii) $$\sum_{k=1}^{\infty} \frac{H_k}{(2k-1)^2} = \tfrac{3}{2}\chi_3(1),$$

(iii) $$\sum_{k=1}^{\infty} \frac{h_k}{k^2} = 2\chi_3(1),$$

and

(iv) $$\sum_{k=1}^{\infty} \frac{h_k(\sqrt{5}-2)^{2k-1}}{2k-1} = \frac{\pi^2}{60} + \tfrac{3}{4}\operatorname{Log}^2\left(\frac{\sqrt{5}-1}{2}\right) + (\sqrt{5}+2)\operatorname{Log} 4 + (3\sqrt{5}+5+\operatorname{Log} 2)\operatorname{Log}\left(\frac{\sqrt{5}-1}{2}\right).$$

Proof of (i). For $|x| \leq 1$, define

$$t(x) = \sum_{k=1}^{\infty} \frac{H_k x^k}{k^2}.$$

Observe that

$$t(x) = \mathrm{Li}_3(x) + g(x), \tag{12.1}$$

where g is defined by (9.1). We wish to evaluate $t(\tfrac{1}{2})$. Using (12.1), Entry 9(i), and Example 6(i), we find that

$$\begin{aligned} t(\tfrac{1}{2}) &= \mathrm{Li}_3(\tfrac{1}{2}) + g(\tfrac{1}{2}) \\ &= \mathrm{Li}_3(\tfrac{1}{2}) - \tfrac{1}{2}\operatorname{Log}^3 2 - \mathrm{Li}_2(\tfrac{1}{2})\operatorname{Log} 2 - \mathrm{Li}_3(\tfrac{1}{2}) + \zeta(3) \\ &= -\tfrac{1}{2}\operatorname{Log}^3 2 - \left(\frac{\pi^2}{12} - \tfrac{1}{2}\operatorname{Log}^2 2\right)\operatorname{Log} 2 + \zeta(3), \end{aligned}$$

from which the desired result follows.

Proof of (ii). We wish to evaluate $\{t(1) - t(-1)\}/2$. By (12.1), (9.4), and Entry 9(iv),

$$\begin{aligned} \tfrac{1}{2}\{t(1) - t(-1)\} &= \tfrac{1}{2}\{g(1) - g(-1)\} + \tfrac{1}{2}\{\mathrm{Li}_3(1) - \mathrm{Li}_3(-1)\} \\ &= \tfrac{1}{2}\{\zeta(3) - \tfrac{1}{2}(\mathrm{Li}_3(-1) + \zeta(3))\} + \chi_3(1) \\ &= \tfrac{1}{2}\chi_3(1) + \chi_3(1) = \tfrac{3}{2}\chi_3(1). \end{aligned}$$

Proof of (iii). The left side of (iii) is $4F(1)$, where F is defined in Entry 11. Putting $x = 0$ in Entry 11(i), we find that $F(1) = \chi_3(1)/2$. Hence, the result follows.

Proof of (iv). The left side of (iv) is $H(\sqrt{5}-2)$, where H is defined in Entry 12. Putting $x = (\sqrt{5}-1)/2$ in Entry 12 and noting that $(1-x)/(1+x) = \sqrt{5}-2$, we find that

$$\begin{aligned} H(\sqrt{5}-2) = {} & (\operatorname{Log} 2 - 1)\operatorname{Log}\left(\frac{\sqrt{5}-1}{2}\right) + 3(\sqrt{5}+2)\operatorname{Log}\left(\frac{\sqrt{5}-1}{2}\right) \\ & + 2(\sqrt{5}+2)\operatorname{Log} 2 + \tfrac{1}{4}\operatorname{Log}^2\left(\frac{\sqrt{5}-1}{2}\right) \\ & + \frac{\pi^2}{12} + \mathrm{Li}_2\left(-\frac{\sqrt{5}-1}{2}\right). \end{aligned} \tag{12.2}$$

Since, by (6.2), $\mathrm{Li}_2(-x) = \mathrm{Li}_2(x) - 2\chi_2(x)$, we find from Examples 6(ii) and (v) that

$$\mathrm{Li}_2\left(-\frac{\sqrt{5}-1}{2}\right) = -\frac{\pi^2}{15} + \tfrac{1}{2}\,\mathrm{Log}^2\left(\frac{\sqrt{5}-1}{2}\right).$$

Using this value in (12.2) and simplifying, we deduce the sought result.

Example (ii) was first established by Nielsen [4]. Jordan [1], [2] apparently not only first proved Example (iii) but also found a general formula for $\sum_{k=1}^{\infty} h_k/k^{2n}$, where n is a positive integer. Later proofs of both (ii) and (iii) were found by Sitaramachandrarao and Siva Rama Sarma [2].

For other formulas like those in Sections 9 and 12, consult the papers of Nielsen [1], [2], [3], [4] (as well as his book [6]); Euler [2], [8]; Jordan [1], [2]; Gates, Gerst, and Kac [1]; Schaeffer [1]; Gupta [1]; Hans and Dumir [1]; Sitaramachandrarao and Siva Rama Sarma [1] (as well as Siva Rama Sarma's thesis [1]); Sitaramachandrarao and Subbarao [1], [2]; Buschman [1]; Rutledge and Douglass [1]; Georghiou and Philippou [1]; and Klamkin [1], [2].

Closely connected with the polylogarithms are the Clausen functions $\mathrm{Cl}_n(x)$ defined by (see Lewin [1, p. 191])

$$\mathrm{Cl}_{2n}(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^{2n}}, \qquad n \geq 1, \tag{13.1}$$

and

$$\mathrm{Cl}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2n+1}}, \qquad n \geq 0, \tag{13.2}$$

where x is real, with the restriction that x is not a multiple of 2π when $n = 0$. It should be noted that

$$\mathrm{Cl}_1(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k} = -\mathrm{Log}\left|2\sin\left(\frac{x}{2}\right)\right|. \tag{13.3}$$

Entry 13. *If n is a positive integer, then*

$$\int_0^x \frac{u^n}{2}\cot\left(\frac{u}{2}\right) du$$
$$= \cos\left(\frac{n\pi}{2}\right) n!\,\zeta(n+1) - \sum_{j=0}^{n} (-1)^{j(j+1)/2} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} x^{n-j}\mathrm{Cl}_{j+1}(x).$$

Proof. For each positive integer k, we have upon n integrations by parts

$$\int_0^x u^n \sin(ku)\,du = -\frac{x^n}{k}\cos(kx) + \frac{nx^{n-1}}{k^2}\sin(kx)$$
$$+ \frac{n(n-1)x^{n-2}}{k^3}\cos(kx) - \frac{n(n-1)(n-2)x^{n-3}}{k^4}\sin(kx)$$
$$+ \cdots + f_n(x) + \frac{n!}{k^{n+1}}\cos\left(\frac{n\pi}{2}\right), \tag{13.4}$$

where $f_n(x) = (-1)^{m+1} n! \cos(kx)/k^{n+1}$, if $n = 2m$ is even, and

$$f_n(x) = (-1)^m n! \sin(kx)/k^{n+1},$$

if $n = 2m + 1$ is odd. Now sum both sides of (13.4) on k, $1 \le k \le N$, and let N tend to ∞ to get (see Gradshteyn and Ryzhik [1, p. 30])

$$\lim_{N\to\infty} \int_0^x \frac{u^n}{2}\left\{\cos\left(\frac{u}{2}\right) - \cos(N+\tfrac{1}{2})u\right\} \csc\left(\frac{u}{2}\right) du$$

$$= \int_0^x \frac{u^n}{2} \cot\left(\frac{u}{2}\right) du = -\sum_{j=0}^{n} (-1)^{j(j+1)/2} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} x^{n-j} \mathrm{Cl}_{j+1}(x)$$

$$+ n!\,\zeta(n+1)\cos\left(\frac{n\pi}{2}\right),$$

where we have used the Riemann-Lebesgue lemma.

Entry 13 is equivalent to formulas in Lewin's book [1, p. 200, equations (7.52) and (7.53)].

Next, define

$$D_{2n}(x) = \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{(2k-1)^{2n}}, \qquad n \ge 1, \tag{14.1}$$

and

$$D_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^{2n+1}}, \qquad n \ge 0, \tag{14.2}$$

where x is real, with the restriction that x is not a multiple of π when $n = 0$. Observe that (Gradshteyn and Ryzhik [1, p. 38])

$$D_1(x) = -\tfrac{1}{2} \operatorname{Log}\left|\tan\left(\frac{x}{2}\right)\right|. \tag{14.3}$$

Entry 14. *If n is a positive integer, then*

$$\int_0^x \frac{u^n}{2} \csc u\, du = \cos\left(\frac{n\pi}{2}\right) n!\, \chi_{n+1}(1)$$

$$- \sum_{j=0}^{n} (-1)^{j(j+1)/2} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} x^{n-j} D_{j+1}(x), \tag{14.4}$$

where χ_n is defined by (6.2).

Proof. The proof follows along the same lines as that of Entry 13. We begin with (13.4) but with k replaced by $2k-1$. Now sum on k, $1 \le k \le N$, and let N tend to ∞. It is easily seen that we get the right side of (14.4). On the left side, we obtain (Gradshteyn and Ryzhik [1, p. 30])

$$\lim_{N\to\infty} \int_0^x \frac{u^n}{2}\{1 - \cos(2Nu)\} \csc u\, du = \int_0^x \frac{u^n}{2} \csc u\, du,$$

by the Riemann–Lebesgue lemma. This finishes the proof.

On the right side of (14.4), Ramanujan has written $\mathrm{Li}_{n+1}(1)$ instead of $\chi_{n+1}(1)$, p. 109.

Entry 15. *For each nonnegative integer n, define*

$$f_n(x) = \int x^n \cot x\, dx.$$

Then if $n \geq 0$,

$$2^n f_n\left(\frac{\pi}{2} - x\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \pi^{n-k} \{f_k(2x) - 2^k f_k(x)\}.$$

Proof. Now, since $\tan x = \cot x - 2\cot(2x)$,

$$\begin{aligned}
2^n f_n\left(\frac{\pi}{2} - x\right) &= 2^n \int \left(\frac{\pi}{2} - x\right)^n \cot\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right) \\
&= -2^n \int \left(\frac{\pi}{2} - x\right)^n \tan x\, dx \\
&= -2^n \int \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\pi}{2}\right)^{n-k} (-x)^k \{\cot x - 2\cot(2x)\}\, dx \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \pi^{n-k} \left(\int (2x)^k \cot(2x)\, d(2x) - \int (2x)^k \cot x\, dx \right) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \pi^{n-k} \{f_k(2x) - 2^k f_k(x)\}.
\end{aligned}$$

Examples. *We have*

$$\int x^n \cot x\, dx = \int \frac{(\sin^{-1} y)^n}{y}\, dy \tag{15.1}$$

and

$$\int \frac{2x^n}{\sin(2x)}\, dx = \int \frac{(\tan^{-1} z)^n}{z}\, dz. \tag{15.2}$$

Proof. Equality (15.1) arises from setting $y = \sin x$, and (15.2) is gotten by letting $z = \tan x$.

Recall that h_n is defined by (8.2).

Proposition 15. *For* $|x| \leq 1$,

(i) $$\tfrac{1}{2}(\tan^{-1} x)^2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k x^{2k}}{2k},$$

(ii) $$\tfrac{1}{2}(\sin^{-1} x)^2 = \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2 x^{2k+2}}{(2k+2)!},$$

(iii) $$\frac{1}{3!}(\sin^{-1} x)^3 = \sum_{k=1}^{\infty} \left(1 + \frac{1}{3^2} + \cdots + \frac{1}{(2k-1)^2}\right)\binom{2k}{k}\frac{x^{2k+1}}{2^{2k}(2k+1)},$$

and

(iv) $$\frac{1}{4!}(\sin^{-1} x)^4 = \sum_{k=1}^{\infty} \left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{(2k)^2}\right)\frac{2^{2k}(k!)^2 x^{2k+2}}{(2k+2)!}.$$

Proof. The Maclaurin series (i)–(iii) may be found in J. Edwards' calculus book [1, pp. 88–90] where the methods for deriving them are clearly delineated. Since (iv) is not given, we shall prove (iv).

Write, for $|x| \leq 1$,

$$y = e^{a \sin^{-1} x} = \sum_{k=0}^{\infty} a_k x^k. \tag{15.3}$$

Then $y' = ay/\sqrt{1-x^2}$, or

$$(1-x^2)(y')^2 = a^2 y^2.$$

Differentiate both sides above with respect to x and then divide both sides by $2y'$ to obtain

$$(1-x^2)y'' - xy' - a^2 y = 0.$$

Substituting the power series (15.3) into the differential equation above and equating coefficients of like powers of x, we find that

$$a_{k+2} = \frac{(k^2+a^2)a_k}{(k+2)(k+1)}, \qquad k \geq 0.$$

Moreover, it is easily seen that $a_0 = 1$ and $a_1 = a$. A simple inductive argument now gives

$$a_{2n+1} = \frac{a(a^2+1^2)(a^2+3^2)\cdots(a^2+(2n-1)^2)}{(2n+1)!}, \qquad n \geq 1, \tag{15.4}$$

and

$$a_{2n} = \frac{a^2(a^2+2^2)(a^2+4^2)\cdots(a^2+(2n-2)^2)}{(2n)!}, \qquad n \geq 1. \tag{15.5}$$

Expanding $\exp(a \sin^{-1} x)$ as a power series in a, and equating coefficients of a^k on both sides, with the use of (15.4) and (15.5), we may deduce the Maclaurin series for $(\sin^{-1} x)^k$, $k \geq 1$. In particular, for $k = 4$, we find from (15.5) that

$$\frac{(\sin^{-1} x)^4}{4!} = \sum_{n=2}^{\infty} \frac{b_{2n} x^{2n}}{(2n)!},$$

where

$$b_{2n} = \sum_{j=1}^{n-1} \frac{2^2 4^2 \cdots (2n-2)^2}{(2j)^2} = 2^{2(n-1)}\{(n-1)!\}^2 \sum_{j=1}^{n-1} \frac{1}{(2j)^2},$$

which completes the proof of (iv).

Entry 16. *For* $|x| \le \pi/2$,

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{\sin^{2k+1} x}{2^{2k}(2k+1)^2} = x \operatorname{Log}|2 \sin x| + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin(2kx)}{k^2}. \tag{16.1}$$

Proof. Making the substitution $t = \sin(u/2)$ and employing Entry 13 with $n = 1$, we find, for $|x| \le \pi/2$, that

$$\int_0^{\sin x} \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(t/2)^{2k}}{2k+1} dt = \int_0^{\sin x} \frac{\sin^{-1} t}{t} dt = \frac{1}{4} \int_0^{2x} u \cot\left(\frac{u}{2}\right) du$$

$$= \tfrac{1}{2}\{-2x \operatorname{Cl}_1(2x) + \operatorname{Cl}_2(2x)\}. \tag{16.2}$$

If we now use (13.1) and (13.3) in (16.2), we deduce (16.1).

Let

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}; \tag{16.3}$$

thus, G denotes Catalan's constant.

Examples. *We have*

(i) $$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{2k}(2k+1)^2} = \frac{\pi}{2} \operatorname{Log} 2,$$

(ii) $$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{3k}(2k+1)^2} = \frac{\pi}{4\sqrt{2}} \operatorname{Log} 2 + \frac{1}{\sqrt{2}} G,$$

(iii) $$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{4k+1}(2k+1)^2} = \frac{3\sqrt{3}}{4} \sum_{k=0}^{\infty} \frac{1}{(3k+1)^2} - \frac{\pi^2}{6\sqrt{3}},$$

and

(iv) $$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{3^k}{2^{4k}(2k+1)^2} = \frac{\pi}{3\sqrt{3}} \operatorname{Log} 3 - \frac{2\pi^2}{27} + \sum_{k=0}^{\infty} \frac{1}{(3k+1)^2}.$$

Proof. Part (i) follows from putting $x = \pi/2$ in (16.1).

Put $x = \pi/4$ in (16.1) and multiply the resulting equality by $\sqrt{2}$. We then obtain (ii).

Next, let $x = \pi/6$ in (16.1). The left side of (16.1) becomes the left side of (iii), and the right side of (16.1) is found to be

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin(k\pi/3)}{k^2}$$

$$= \frac{3\sqrt{3}}{4} \sum_{k=0}^{\infty} \frac{1}{(3k+1)^2} + \frac{\sqrt{3}}{4}\left\{-2 \sum_{k=0}^{\infty} \frac{1}{(6k+1)^2} + \sum_{k=0}^{\infty} \frac{1}{(6k+2)^2} - 4 \sum_{k=0}^{\infty} \frac{1}{(6k+4)^2} - \sum_{k=0}^{\infty} \frac{1}{(6k+5)^2}\right\}$$

$$=\frac{3\sqrt{3}}{4}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2}-\frac{\sqrt{3}}{2}\left\{\sum_{k=0}^{\infty}\frac{1}{(6k+1)^2}+\sum_{k=0}^{\infty}\frac{1}{(6k+5)^2}\right\}$$

$$=\frac{3\sqrt{3}}{4}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2}-\frac{\sqrt{3}}{2}\left\{\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}-\frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\right\}$$

$$=\frac{3\sqrt{3}}{4}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2}-\frac{4\sqrt{3}}{9}\frac{\pi^2}{8},$$

which completes the proof of (iii).

Lastly, put $x=\pi/3$ in (16.1) and multiply the resulting equality by $2/\sqrt{3}$. The left side then becomes the left side of (iv), and the right side is equal to

$$\frac{\pi}{3\sqrt{3}}\operatorname{Log}3+\frac{1}{\sqrt{3}}\sum_{k=1}^{\infty}\frac{\sin(2k\pi/3)}{k^2}$$
$$=\frac{\pi}{3\sqrt{3}}\operatorname{Log}3+\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2}-\frac{1}{2}\left\{\sum_{k=1}^{\infty}\frac{1}{k^2}-\sum_{k=1}^{\infty}\frac{1}{(3k)^2}\right\},$$

which is readily seen to equal the right side of (iv).

Part (i) can be found in a paper of Ruscheweyh [1].

Entry 17. *For* $|x|<\pi/2$,

$$\sum_{k=0}^{\infty}\frac{(-1)^k\tan^{2k+1}x}{(2k+1)^2}=x\operatorname{Log}|\tan x|+\sum_{k=0}^{\infty}\frac{\sin(4k+2)x}{(2k+1)^2}. \tag{17.1}$$

Proof. For $|x|<\pi/2$, the left side of (17.1) is equal to

$$\int_0^{\tan x}\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{2k+1}\,dt=\int_0^{\tan x}\frac{\tan^{-1}t}{t}\,dt=\frac{1}{2}\int_0^{2x}u\csc u\,du$$
$$=-2xD_1(2x)+D_2(2x), \tag{17.2}$$

where in the penultimate step we made the substitution $u=2\tan^{-1}t$, and in the last step we utilized Entry 14 with $n=1$. If we now employ (14.1) and (14.3), we deduce (17.1).

Examples. *We have*

(i) $$\int_0^{1/\sqrt{3}}\frac{\tan^{-1}t}{t}\,dt=-\frac{\pi}{12}\operatorname{Log}3-\frac{5\pi^2}{18\sqrt{3}}+\frac{5\sqrt{3}}{4}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2},$$

(ii) $$\int_0^{\sqrt{2}-1}\frac{\tan^{-1}t}{t}\,dt=\frac{\pi}{8}\operatorname{Log}(\sqrt{2}-1)-\frac{\pi^2}{16}+\sqrt{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2},$$

and

(iii) $$\int_0^{2-\sqrt{3}} \frac{\tan^{-1} t}{t}\,dt = \frac{\pi}{12}\operatorname{Log}(2-\sqrt{3}) + \frac{2}{3}\int_0^1 \frac{\tan^{-1} t}{t}\,dt,$$

where

$$\int_0^1 \frac{\tan^{-1} t}{t}\,dt = G = 0.915965594177\ldots,$$

where G is defined by (16.3).

Proof of (i). In Entry 17, put $x = \pi/6$ and use (17.2) to obtain

$$\int_0^{1/\sqrt{3}} \frac{\tan^{-1} t}{t}\,dt = -\frac{\pi}{12}\operatorname{Log} 3 + \frac{\sqrt{3}}{2}\left(1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \cdots\right)$$

$$= -\frac{\pi}{12}\operatorname{Log} 3 + \frac{\sqrt{3}}{2}\left\{\left(1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{10^2} - \frac{1}{11^2} + \cdots\right)\right.$$
$$\left. + \left(\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{8^2} - \frac{1}{10^2} + \cdots\right)\right\}$$

$$= -\frac{\pi}{12}\operatorname{Log} 3 + \frac{5\sqrt{3}}{8}\left(1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{10^2} - \frac{1}{11^2} + \cdots\right)$$

$$= -\frac{\pi}{12}\operatorname{Log} 3 + \frac{5\sqrt{3}}{4}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2} - \frac{5\sqrt{3}}{4}\left\{\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{\infty}\frac{1}{(3k)^2}\right\},$$

from which (i) follows.

Proof of (ii). Set $x = \pi/8$ in Entry 17. Using (17.2) and the fact that $\tan(\pi/8) = \sqrt{2} - 1$, we find that

$$\int_0^{\sqrt{2}-1} \frac{\tan^{-1} t}{t}\,dt = \frac{\pi}{8}\operatorname{Log}(\sqrt{2}-1) + \sum_{k=0}^{\infty}\frac{\sin\{(2k+1)\pi/4\}}{(2k+1)^2}$$

$$= \frac{\pi}{8}\operatorname{Log}(\sqrt{2}-1) + \frac{2}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2}$$
$$- \frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{(-1)^{k(k+1)/2}}{(2k+1)^2}.$$

The latter sum in the equality above is $L(2, \chi)$, where χ is the real, even, primitive character of modulus 8. By a standard formula (see, e.g., Berndt and Schoenfeld's paper [1, p. 48]), $L(2, \chi) = \pi^2\sqrt{2}/16$. This completes the proof of (ii).

Proof of (iii). Let $x = \pi/12$ in Entry 17. Noting that $\tan(\pi/12) = 2 - \sqrt{3}$ and using (17.2), we find that

$$\int_0^{2-\sqrt{3}} \frac{\tan^{-1} t}{t}\, dt = \frac{\pi}{12} \operatorname{Log}(2-\sqrt{3}) + \sum_{k=0}^{\infty} \frac{\sin\{(2k+1)\pi/6\}}{(2k+1)^2}$$

$$= \frac{\pi}{12} \operatorname{Log}(2-\sqrt{3}) + \tfrac{1}{2}\left(1 + \frac{1}{5^2} - \frac{1}{7^2} - \frac{1}{11^2} + \cdots\right)$$

$$+ \frac{1}{3^2} - \frac{1}{9^2} + \frac{1}{15^2} - \frac{1}{21^2} + \cdots$$

$$= \frac{\pi}{12} \operatorname{Log}(2-\sqrt{3})$$

$$+ \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(6k+3)^2}\right) + \tfrac{1}{9}G$$

$$= \frac{\pi}{12} \operatorname{Log}(2-\sqrt{3}) + \tfrac{5}{9}G + \tfrac{1}{9}G.$$

The given integral representation for G follows easily upon integrating the Maclaurin series for $(\tan^{-1} t)/t$ termwise, and so the proof of (iii) is completed.

The decimal expansion of G is correct to the number of places given. In fact, this decimal expansion for G can be found in J. Edwards' book [2, p. 246], a text with which Ramanujan was very familiar.

Entry 17′ as well as Examples (ii) and (iii) may be found in Ramanujan's paper [10], [15, pp. 40–43].

Entry 18. *For* $0 \le x \le \pi/4$,

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{\cos^{2k+1} x - \sin^{2k+1} x}{2^{2k}(2k+1)^2} = \frac{\pi}{2} \operatorname{Log}(2\cos x)$$

$$- \frac{1}{2} \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\sin^{2k+1}(2x)}{2^{2k}(2k+1)^2}. \quad (18.1)$$

Proof. Replacing x by $\pi/2 - x$ in Entry 16, we find that, for $0 \le x \le \pi$,

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{\cos^{2k+1} x}{2^{2k}(2k+1)^2} = \left(\frac{\pi}{2} - x\right) \operatorname{Log}|2\cos x|$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sin(2kx)}{k^2}. \quad (18.2)$$

Subtracting (16.1) from (18.2), we deduce that, for $0 \le x \le \pi/2$,

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{\cos^{2k+1} x - \sin^{2k+1} x}{2^{2k}(2k+1)^2} = \frac{\pi}{2} \operatorname{Log}(2\cos x) - x \operatorname{Log}(2\sin(2x)) - \sum_{k=1}^{\infty} \frac{\sin(4kx)}{(2k)^2}.$$

Replacing x by $2x$ in Entry 16, we get, for $|x| \le \pi/4$,

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{\sin^{2k+1}(2x)}{2^{2k}(2k+1)^2} = 2x \operatorname{Log}|2\sin(2x)| + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin(4kx)}{k^2}.$$

Combining the latter two equalities, we deduce (18.1) for $0 \le x \le \pi/4$.

Example. *For* $|x| \le 1$, *define*

$$\psi(x) = \int_0^x \frac{\sin^{-1} t}{t}\, dt.$$

Then

$$\psi(\tfrac{3}{5}) - \tfrac{1}{2}\psi(\tfrac{24}{25}) = \frac{\pi}{2} \operatorname{Log} 2 + 2\psi\left(\frac{1}{\sqrt{5}}\right) - 2\psi\left(\frac{2}{\sqrt{5}}\right).$$

Proof. From (16.1) and (16.2), for $|x| \le \pi/2$,

$$\psi(\sin x) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\sin^{2k+1} x}{2^{2k}(2k+1)^2}. \tag{18.3}$$

Let $\sin x = \frac{3}{5}$; so $\cos x = \frac{4}{5}$. Then $\sin(2x) = \frac{24}{25}$. Thus, by Entry 18 and (18.3),

$$\psi(\tfrac{4}{5}) - \psi(\tfrac{3}{5}) = \frac{\pi}{2} \operatorname{Log}(\tfrac{8}{5}) - \tfrac{1}{2}\psi(\tfrac{24}{25}). \tag{18.4}$$

Secondly, let $\sin x = 1/\sqrt{5}$; so $\cos x = 2/\sqrt{5}$ and $\sin(2x) = \frac{4}{5}$. Again, from Entry 18 and (18.3),

$$\psi\left(\frac{2}{\sqrt{5}}\right) - \psi\left(\frac{1}{\sqrt{5}}\right) = \frac{\pi}{2} \operatorname{Log}\left(\frac{4}{\sqrt{5}}\right) - \tfrac{1}{2}\psi(\tfrac{4}{5})$$

$$= \frac{\pi}{4} \operatorname{Log}(\tfrac{8}{5}) + \frac{\pi}{4} \operatorname{Log} 2 - \tfrac{1}{2}\psi(\tfrac{4}{5}). \tag{18.5}$$

Combining (18.4) and (18.5) together, we deduce the proposed equality.

Entry 19. *For* $0 \le x < \pi/2$,

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{\cos^{2k+1} x + \sin^{2k+1} x}{2^{2k}(2k+1)^2} = \frac{\pi}{2} \operatorname{Log}(2\cos x) + \sum_{k=0}^{\infty} \frac{(-1)^k \tan^{2k+1} x}{(2k+1)^2}.$$

Proof. Adding (16.1) and (18.2), we find that, for $0 \leq x < \pi/2$,

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{\cos^{2k+1} x + \sin^{2k+1} x}{2^{2k}(2k+1)^2} = \frac{\pi}{2} \operatorname{Log}(2 \cos x) + x \operatorname{Log}(\tan x) + \sum_{k=0}^{\infty} \frac{\sin(4k+2)x}{(2k+1)^2}$$

$$= \frac{\pi}{2} \operatorname{Log}(2 \cos x) + \sum_{k=0}^{\infty} \frac{(-1)^k \tan^{2k+1} x}{(2k+1)^2},$$

by Entry 17.

Example. *We have*

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1 + 2^{2k+1}}{2^{2k} 5^{k+1} (2k+1)^2} = \frac{\pi}{2\sqrt{5}} \operatorname{Log}\left(\frac{4}{\sqrt{5}}\right) + \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+1}(2k+1)^2}.$$

Proof. In Entry 19, put $\sin x = 1/\sqrt{5}$; so $\cos x = 2/\sqrt{5}$ and $\tan x = \frac{1}{2}$. The proposed formula now follows.

Entry 20. *Let* $|x| \leq \pi/2$. *Then*

$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2 \sin^{2k+2} x}{(2k+1)!\,(2k+2)^2} = \frac{x^2}{2} \operatorname{Log}|2 \sin x| + \frac{x}{2} \sum_{k=1}^{\infty} \frac{\sin(2kx)}{k^2} + \frac{1}{4} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k^3} - \tfrac{1}{4}\zeta(3). \tag{20.1}$$

Proof. By Proposition 15(ii), the left side of (20.1), for $|x| \leq \pi/2$, is equal to

$$\int_0^{\sin x} \frac{(\sin^{-1} t)^2}{2t} \, dt = \frac{1}{8} \int_0^{2x} \frac{u^2}{2} \cot\left(\frac{u}{2}\right) du$$

$$= \tfrac{1}{8}\{-2\zeta(3) - 4x^2 \operatorname{Cl}_1(2x) + 4x \operatorname{Cl}_2(2x) + 2 \operatorname{Cl}_3(2x)\}.$$

In the first equality, we made the substitution $u = 2 \sin^{-1} t$, and to get the last equality we used Entry 13 with $n = 2$. If we now employ (13.1)–(13.3) in the equality above, we deduce (20.1).

In the notebooks, p. 111, the term $-\zeta(3)/4$ in (20.1) has been omitted.

Examples. *We have*

(i) $$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(2k+1)!\,(2k+2)^2} = \frac{\pi^2}{8} \operatorname{Log} 2 - \tfrac{1}{2}\chi_3(1)$$

and

(ii) $$\sum_{k=0}^{\infty} \frac{2^{k-1}(k!)^2}{(2k+1)!\,(2k+2)^2} = \frac{\pi^2}{64} \operatorname{Log} 2 + \frac{\pi}{8} G - \tfrac{5}{16}\chi_3(1),$$

where χ_3 *is defined by* (6.2) *and* G *is defined by* (16.3).

Proof. To obtain (i), simply set $x=\pi/2$ in Entry 20.

Putting $x=\pi/4$ in Entry 20, we see immediately that the left side of (20.1) yields the left side of (ii). On the right side, we get

$$\frac{\pi^2}{64}\operatorname{Log} 2+\frac{\pi}{8}G+\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)^3}-\tfrac{1}{4}\zeta(3).$$

The last two expressions are together equal to $-35\zeta(3)/128=-5\chi_3(1)/16$, and so the proof is complete.

Entry 21. *For* $|x|\le\pi/4$,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}h_k}{(2k)^2}\tan^{2k}x=\frac{x^2}{2}\operatorname{Log}|\tan x|+x\sum_{k=0}^{\infty}\frac{\sin(4k+2)x}{(2k+1)^2}$$
$$+\frac{1}{2}\sum_{k=0}^{\infty}\frac{\cos(4k+2)x}{(2k+1)^3}-\tfrac{1}{2}\chi_3(1), \tag{21.1}$$

where h_k *is defined in* (8.2).

Proof. Using Proposition 15(i), then making the substitution $u=2\tan^{-1}t$, and finally employing Entry 14 with $n=2$, we find that the left side of (21.1) is equal to, for $|x|\le\pi/4$,

$$\int_0^{\tan x}\frac{(\tan^{-1}t)^2}{2t}\,dt=\frac{1}{4}\int_0^{2x}\frac{u^2}{2}\csc u\,du$$
$$=\tfrac{1}{4}\{-2\chi_3(1)-4x^2D_1(2x)+4xD_2(2x)+2D_3(2x)\}.$$

Using (14.1)–(14.3) above, we complete the proof.

Example. *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}h_k}{k^2}=\pi G-2\chi_3(1), \tag{21.2}$$

where G denotes Catalan's constant.

Proof. Let $x=\pi/4$ in (21.1) and multiply the resulting equality by 4 to achieve (21.2).

In the notebooks, p. 112, the right side of (21.2) is incorrectly multiplied by $\frac{1}{4}$. Nielsen [4] evidently first established (21.2).

Entry 22. *Let* $0\le x\le\pi/2$. *Then*

$$\sum_{k=0}^{\infty}\frac{2^{2k}(k!)^2\{\cos^{2k+2}x+\sin^{2k+2}x\}}{(2k+1)!\,(2k+2)^2}=-\frac{\pi^2}{8}\operatorname{Log}(2\cos x)$$
$$+\frac{\pi}{2}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\cos^{2k+1}x}{2^{2k}(2k+1)^2}+\frac{1}{4}\sum_{k=0}^{\infty}\frac{2^{2k}(k!)^2\sin^{2k+2}(2x)}{(2k+1)!\,(2k+2)^2}-\tfrac{1}{2}\chi_3(1).$$

Proof. Replacing x by $\pi/2 - x$ in Entry 20, we find that, for $0 \le x \le \pi$,

$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2 \cos^{2k+2} x}{(2k+1)!\,(2k+2)^2} = \frac{1}{2}\left(\frac{\pi^2}{4} - \pi x + x^2\right) \operatorname{Log}|2\cos x| + \frac{1}{2}\left(\frac{\pi}{2} - x\right) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\sin(2kx)}{k^2} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{(-1)^k \cos(2kx)}{k^3} - \tfrac{1}{4}\zeta(3). \tag{22.1}$$

Adding (20.1) and (22.1), we deduce that, for $0 \le x \le \pi/2$,

$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2\{\cos^{2k+2} x + \sin^{2k+2} x\}}{(2k+1)!\,(2k+2)^2} = -\frac{\pi^2}{8}\operatorname{Log}(2\cos x) + \frac{\pi}{2}\left\{\left(\frac{\pi}{2} - x\right)\operatorname{Log}(2\cos x) + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin(2kx)}{k^2}\right\} + \tfrac{1}{4}\left\{2x^2\operatorname{Log}|2\sin(2x)| + x\sum_{k=1}^{\infty}\frac{\sin(4kx)}{k^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{\cos(4kx)}{k^3} - \tfrac{1}{4}\zeta(3)\right\} - \tfrac{7}{16}\zeta(3). \tag{22.2}$$

Observe that the former expression in curly brackets on the right side of (22.2) is equal to the right side of (18.2). Secondly, note that the latter expression in curly brackets on the right side of (22.2) is equal to the right side of (20.1), but with x replaced by $2x$. Lastly, note that $7\zeta(3)/8 = \chi_3(1)$. Employing all of these observations, we see that (22.2) reduces to the desired equality.

Entry 23. *For* $|x| \le \pi/4$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}h_k}{(2k)^2}\tan^{2k} x = 2\sum_{k=0}^{\infty}\frac{2^{2k}(k!)^2\sin^{2k+2} x}{(2k+1)!\,(2k+2)^2} - \frac{1}{4}\sum_{k=0}^{\infty}\frac{2^{2k}(k!)^2\sin^{2k+2}(2x)}{(2k+1)!\,(2k+2)^2}, \tag{23.1}$$

where h_k *is defined in* (8.2).

Proof. By Entry 20, the right side of (23.1) is equal to, for $|x| \le \pi/4$,

$$2\left\{\frac{x^2}{2}\operatorname{Log}|2\sin x| + \frac{x}{2}\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k^3} - \tfrac{1}{4}\zeta(3)\right\} - \tfrac{1}{4}\left\{2x^2\operatorname{Log}|2\sin(2x)| + x\sum_{k=1}^{\infty}\frac{\sin(4kx)}{k^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{\cos(4kx)}{k^3} - \tfrac{1}{4}\zeta(3)\right\}$$

$$= \frac{x^2}{2}\operatorname{Log}|\tan x| + x\sum_{k=0}^{\infty}\frac{\sin(4k+2)x}{(2k+1)^2} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{\cos(4k+2)x}{(2k+1)^3} - \tfrac{1}{2}\chi_3(1).$$

Entry 21 now implies (23.1).

Entry 24. *Let x, y, θ, and φ be real numbers with* $xe^{i\theta}+ye^{i\varphi}=1$, $0\le x,y\le 1$, *and* $-\pi<\theta,\varphi\le\pi$. *Then*

(i) $$\sum_{k=1}^{\infty}\frac{x^k\cos(k\theta)}{k^2}+\sum_{k=1}^{\infty}\frac{y^k\cos(k\varphi)}{k^2}=\frac{\pi^2}{6}-\operatorname{Log}x\operatorname{Log}y+\theta\varphi$$

and

(ii) $$\sum_{k=1}^{\infty}\frac{x^k\sin(k\theta)}{k^2}+\sum_{k=1}^{\infty}\frac{y^k\sin(k\varphi)}{k^2}=-\varphi\operatorname{Log}x-\theta\operatorname{Log}y.$$

Proof. Using Entry 6(iii) below, we find that

$$\sum_{k=1}^{\infty}\frac{x^ke^{ik\theta}}{k^2}+\sum_{k=1}^{\infty}\frac{y^ke^{ik\varphi}}{k^2}=\operatorname{Li}_2(xe^{i\theta})+\operatorname{Li}_2(ye^{i\varphi})$$
$$=\frac{\pi^2}{6}-(\operatorname{Log}x+i\theta)(\operatorname{Log}y+i\varphi).$$

Equating real and imaginary parts on both sides above, we deduce (i) and (ii), respectively.

Entry 25. *Let x, y, θ, and φ be real numbers such that* $xe^{i\theta}+ye^{i\varphi}=xye^{i(\theta+\varphi)}$, $0\le x,y\le 1$, *and* $-\pi<\theta,\varphi\le\pi$. *Then*

(i) $$\sum_{k=1}^{\infty}\frac{x^k\cos(k\theta)}{k^2}+\sum_{k=1}^{\infty}\frac{y^k\cos(k\varphi)}{k^2}$$
$$=\tfrac{1}{8}\operatorname{Log}(1-2x\cos\theta+x^2)\operatorname{Log}(1-2y\cos\varphi+y^2)$$
$$-\tfrac{1}{2}\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)\tan^{-1}\left(\frac{y\sin\varphi}{1-y\cos\varphi}\right)$$

and

(ii) $$\sum_{k=1}^{\infty}\frac{x^k\sin(k\theta)}{k^2}+\sum_{k=1}^{\infty}\frac{y^k\sin(k\varphi)}{k^2}$$
$$=-\tfrac{1}{4}\operatorname{Log}(1-2x\cos\theta+x^2)\tan^{-1}\left(\frac{y\sin\varphi}{1-y\cos\varphi}\right)$$
$$-\tfrac{1}{4}\operatorname{Log}(1-2y\cos\varphi+y^2)\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right).$$

Proof. We shall apply Entry 6(i) with $1-z=xe^{i\theta}$. Then $1-1/z=xe^{i\theta}/(xe^{i\theta}-1)=ye^{i\varphi}$. Since also $1-xe^{i\theta}=1/(1-ye^{i\varphi})$, we find that

$$\operatorname{Li}_2(xe^{i\theta})+\operatorname{Li}_2(ye^{i\varphi})$$
$$=-\tfrac{1}{2}\operatorname{Log}^2(1-xe^{i\theta})=\tfrac{1}{2}\operatorname{Log}(1-xe^{i\theta})\operatorname{Log}(1-ye^{i\varphi})$$
$$=\tfrac{1}{2}\left\{\tfrac{1}{2}\operatorname{Log}(1-2x\cos\theta+x^2)-i\tan^{-1}\left(\frac{y\sin\theta}{1-x\cos\theta}\right)\right\}$$
$$\times\left\{\tfrac{1}{2}\operatorname{Log}(1-2y\cos\varphi+y^2)-i\tan^{-1}\left(\frac{y\sin\varphi}{1-y\cos\varphi}\right)\right\}.$$

Formulas (i) and (ii) now follow from equating real and imaginary parts above, respectively.

Entry 26. *Let x, y, θ, and φ be real numbers satisfying the conditions $xe^{i\theta}+ye^{i\varphi}+xye^{i(\theta+\varphi)}=1$, $0\le x,y\le 1$, and $-\pi<\theta,\varphi\le\pi$. Then*

$$\text{(i)}\quad \sum_{k=0}^{\infty}\frac{x^{2k+1}\cos(2k+1)\theta}{(2k+1)^2}+\sum_{k=0}^{\infty}\frac{y^{2k+1}\cos(2k+1)\varphi}{(2k+1)^2}=\frac{\pi^2}{8}-\tfrac{1}{2}\operatorname{Log}x\operatorname{Log}y+\tfrac{1}{2}\theta\varphi$$

and

$$\text{(ii)}\quad \sum_{k=0}^{\infty}\frac{x^{2k+1}\sin(2k+1)\theta}{(2k+1)^2}+\sum_{k=0}^{\infty}\frac{y^{2k+1}\sin(2k+1)\varphi}{(2k+1)^2}=-\tfrac{1}{2}\varphi\operatorname{Log}x-\tfrac{1}{2}\theta\operatorname{Log}y.$$

Proof. Observe that $ye^{i\varphi}=(1-xe^{i\theta})/(1+xe^{i\theta})$. Thus, Entry 6(v) yields

$$\chi_2(xe^{i\theta})+\chi_2(ye^{i\varphi})=\frac{\pi^2}{8}-\tfrac{1}{2}\operatorname{Log}(xe^{i\theta})\operatorname{Log}(ye^{i\varphi}).$$

Equating real and imaginary parts above, we get (i) and (ii), respectively.

The topic of Sections 27–30 is altogether different from that of the remainder of this chapter, and is a continuation of Ramanujan's studies in Chapter 8. Ramanujan considers

$$\varphi_r(x)=\sum_{k=1}^{x}k^r\operatorname{Log}k, \tag{27.1}$$

where here it is assumed that $r>-1$; in Chapter 8, Ramanujan studied $\varphi_r(x)$ when $r\le-1$ and when $r=-\frac{1}{2}$. In Section 29, Ramanujan examines an analytic function of x which reduces to (27.1) when x is a positive integer. However, Ramanujan does not give any hint at all as to how he has defined his analytic extension of φ_r.

Entry 27(a). *Let $\varphi_r(x)$ be defined by (27.1) for $r>-1$. For each nonnegative integer k, let*

$$M_k(r)=\sum_{j=0}^{k}\frac{1}{r-j}.$$

Then there exists a constant C_r such that as x tends to ∞,

$$\varphi_r(x)-\operatorname{Log}x\left\{\sum_{k=1}^{x}k^r-\zeta(-r)\right\}\sim C_r-\frac{x^{r+1}}{(r+1)^2}+\sum_{k=1}^{\infty}\frac{B_{2k}\Gamma(r+1)M_{2k-2}(r)x^{r-2k+1}}{(2k)!\,\Gamma(r-2k+2)},$$

where B_k denotes the kth *Bernoulli number and where ζ denotes the Riemann zeta-function.*

Proof. We shall apply the Euler–Maclaurin summation formula (I5) to $f(t) = t^r \operatorname{Log} t$. Then as x tends to ∞, we find that

$$\varphi_r(x) \sim \int_1^x t^r \operatorname{Log} t\, dt + \tfrac{1}{2}x^r \operatorname{Log} x + C_r' + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x), \tag{27.2}$$

where C_r' is independent of x.

First, an integration by parts yields

$$\int_1^x t^r \operatorname{Log} t\, dt = \frac{x^{r+1} \operatorname{Log} x}{r+1} - \frac{x^{r+1}}{(r+1)^2} + \frac{1}{(r+1)^2}. \tag{27.3}$$

Secondly, by Leibniz's rule,

$$\begin{aligned} f^{(n)}(t) &= \frac{\Gamma(r+1)}{\Gamma(r+1-n)} t^{r-n} \operatorname{Log} t \\ &\quad + t^{r-n} \sum_{k=0}^{n-1} \binom{n}{k} \frac{\Gamma(r+1)(-1)^{n-k-1}(n-k-1)!}{\Gamma(r+1-k)} \\ &= \frac{\Gamma(r+1)}{\Gamma(r+1-n)} t^{r-n} \operatorname{Log} t + \frac{(-1)^{n+1} n!\, t^{r-n}}{\Gamma(-r)} \sum_{k=0}^{n-1} \frac{\Gamma(-r+k)}{(n-k)k!} \\ &= \frac{\Gamma(r+1)}{\Gamma(r+1-n)} t^{r-n} \operatorname{Log} t + \frac{(-1)^n \Gamma(n-r) M_{n-1}(r) t^{r-n}}{\Gamma(-r)}, \end{aligned} \tag{27.4}$$

by a formula from Hansen's tables [1, p. 126]. Using (27.3) and (27.4) in (27.2), we deduce that

$$\begin{aligned} \varphi_r(x) &\sim \frac{x^{r+1} \operatorname{Log} x}{r+1} - \frac{x^{r+1}}{(r+1)^2} + \frac{x^r \operatorname{Log} x}{2} + C_r \\ &\quad + \sum_{k=1}^{\infty} \frac{B_{2k}\Gamma(r+1)x^{r-2k+1}}{(2k)!\,\Gamma(r+2-2k)} \{\operatorname{Log} x + M_{2k-2}(r)\}, \end{aligned} \tag{27.5}$$

as x tends to ∞, where $C_r = C_r' + 1/(r+1)^2$.

From Entry 1 of Chapter 7, we have

$$\sum_{k=1}^{x} k^r \sim \frac{x^{r+1}}{r+1} + \frac{x^r}{2} + \zeta(-r) + \sum_{k=1}^{\infty} \frac{B_{2k}\Gamma(r+1)x^{r-2k+1}}{(2k)!\,\Gamma(r+2-2k)}, \tag{27.6}$$

as x tends to ∞. Substituting (27.6) into (27.5), we deduce the desired asymptotic formula.

An asymptotic expansion for $\varphi_r(x)$ has also been obtained by MacLeod [1]. For applications of Entry 27(a), see the papers of MacLeod [1] and Ishibashi and Kanemitsu [1].

Entry 27(b). *Let C_r be as in Entry 27(a). Then if $r>0$,*

$$C_r = \frac{2\Gamma(r+1)\zeta(r+1)}{(2\pi)^{r+1}}\left\{\sin\left(\frac{\pi r}{2}\right)\left(\operatorname{Log}(2\pi) - \frac{\Gamma'(r+1)}{\Gamma(r+1)}\right) - \frac{\pi}{2}\cos\left(\frac{\pi r}{2}\right)\right\}$$
$$+ \frac{2\Gamma(r+1)\sin(\pi r/2)}{(2\pi)^{r+1}}\sum_{k=1}^{\infty}\frac{\operatorname{Log} k}{k^{r+1}}.$$

Proof. We shall first show that

$$C_r = -\zeta'(-r). \tag{27.7}$$

It is clear from (27.2) and (27.3) that

$$C_r = \lim_{x\to\infty}\left\{\varphi_r(x) - \frac{x^{r+1}\operatorname{Log} x}{r+1} + \frac{x^{r+1}}{(r+1)^2} - \frac{x^r \operatorname{Log} x}{2}\right.$$
$$\left. - \sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}\frac{d^{2k-1}}{dx^{2k-1}}(x^r \operatorname{Log} x)\right\}, \tag{27.8}$$

where n is chosen so that $n>(r+1)/2$. Applying the Euler–Maclaurin summation formula once again, we find that

$$\varphi_r(x) = \frac{x^{r+1}\operatorname{Log} x}{r+1} - \frac{x^{r+1}}{(r+1)^2} + \frac{1}{(r+1)^2} + \frac{x^r\operatorname{Log} x}{2}$$
$$+ \sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}\left\{\frac{d^{2k-1}}{dx^{2k-1}}(x^r\operatorname{Log} x) - \frac{d^{2k-1}}{dt^{2k-1}}(t^r\operatorname{Log} t)\Big|_{t=1}\right\}$$
$$+ \int_1^x P_{2n+1}(t)\frac{d^{2n+1}}{dt^{2n+1}}(t^r\operatorname{Log} t)\,dt, \tag{27.9}$$

where $P_j(t)$ denotes the jth Bernoulli function. Formulas (27.8) and (27.9) imply that

$$C_r = \frac{1}{(r+1)^2} - \sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}\frac{d^{2k-1}}{dt^{2k-1}}(t^r\operatorname{Log} t)\Big|_{t=1}$$
$$+ \int_1^\infty P_{2n+1}(t)\frac{d^{2n+1}}{dt^{2n+1}}(t^r\operatorname{Log} t)\,dt. \tag{27.10}$$

Now apply the Euler–Maclaurin formula to $f(t)=t^{-s}$, $\operatorname{Re} s>1$, to find that

$$\zeta(s) = \frac{1}{s-1} + \tfrac{1}{2} - \sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}f^{(2k-1)}(t)\Big|_{t=1}$$
$$+ \int_1^\infty P_{2n+1}(t)f^{(2n+1)}(t)\,dt, \tag{27.11}$$

where $n>(r+1)/2$. By analytic continuation, (27.11) holds for $\operatorname{Re} s > -2n$

-1. Differentiating (27.11) and then setting $s = -r$, we find that

$$\zeta'(-r) = \frac{1}{(r+1)^2} + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} \frac{d^{2k-1}}{dt^{2k-1}}(t^r \operatorname{Log} t)\bigg|_{t=1} - \int_1^\infty P_{2n+1}(t) \frac{d^{2n+1}}{dt^{2n+1}}(t^r \operatorname{Log} t)\, dt. \tag{27.12}$$

A comparison of (27.10) and (27.12) yields (27.7).

From the functional equation of $\zeta(s)$, equation (4.2) in Chapter 7, we find that

$$\zeta'(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \left\{ \operatorname{Log}(2\pi) + \frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right) - \frac{\Gamma'(1-s)}{\Gamma(1-s)} - \frac{\zeta'(1-s)}{\zeta(1-s)} \right\}.$$

Putting $s = -r$ yields

$$\zeta'(-r) = -\frac{2\Gamma(r+1)\zeta(r+1)\sin(\pi r/2)}{(2\pi)^{r+1}} \left\{ \operatorname{Log}(2\pi) - \frac{\pi}{2}\cot\left(\frac{\pi r}{2}\right) - \frac{\Gamma'(1+r)}{\Gamma(1+r)} + \frac{1}{\zeta(r+1)} \sum_{k=1}^{\infty} \frac{\operatorname{Log} k}{k^{r+1}} \right\}.$$

By (27.7), the proof is complete.

In the first notebook, p. 163, Ramanujan indicates how he derived Entry 27(b), but his argument is not rigorous.

The following corollary is an immediate consequence of Entry 27(b).

Corollary. *If r is an even positive integer, then*

$$C_r = -\frac{\cos(\pi r/2)\Gamma(r+1)\zeta(r+1)}{2(2\pi)^r}.$$

Ramanujan next records the following particular values of C_r:

$$C_0 = \tfrac{1}{2}\operatorname{Log}(2\pi), \qquad C_2 = \frac{\zeta(3)}{4\pi^2}, \qquad C_4 = -\frac{3\zeta(5)}{4\pi^4},$$

and

$$C_6 = \frac{45\zeta(7)}{8\pi^6}.$$

In the case $r = 0$, we see that C_0 is the constant which occurs in the asymptotic expansion of $\operatorname{Log}\Gamma(x+1)$ (Stirling's formula), and this constant is well known to be $\frac{1}{2}\operatorname{Log}(2\pi)$. (See Entry 23 of Chapter 7.) The values of C_2, C_4, and C_6 are immediately deducible from the Corollary.

The next example is not correctly given by Ramanujan, p. 113. Furthermore, the additive factor of $\frac{1}{4}$ in the denominator of Example (i) may be deleted without affecting the limit.

Example (i). *We have*

$$\lim_{x\to\infty} \frac{\exp(x^2 - \frac{1}{2}\operatorname{Log} x + \frac{1}{2} - \gamma) \prod_{k=1}^{x} k^{2k}}{\left(\frac{x^{x(x+3)}\Gamma(x^2+1)}{x+\frac{1}{4}}\right)^{1/3}} = \prod_{k=1}^{\infty} k^{1/(\pi k)^2}. \tag{27.13}$$

Proof. The logarithm of the left side of (27.13) is

$$L \equiv \lim_{x\to\infty} \left\{2\sum_{k=1}^{x} k \operatorname{Log} k + \tfrac{1}{6}(x^2 - \tfrac{1}{2}\operatorname{Log} x + \tfrac{1}{2} - \gamma)\right.$$
$$\left. - \tfrac{1}{3}x(x+3)\operatorname{Log} x - \tfrac{1}{3}\operatorname{Log}\Gamma(x^2+1) + \tfrac{1}{3}\operatorname{Log}(x+\tfrac{1}{4})\right\}$$
$$= \lim_{x\to\infty} \left\{2\sum_{k=1}^{x} k \operatorname{Log} k + \tfrac{1}{6}(x^2 - \tfrac{1}{2}\operatorname{Log} x + \tfrac{1}{2} - \gamma)\right.$$
$$- \tfrac{1}{3}x(x+3)\operatorname{Log} x - \tfrac{1}{3}((2x^2+1)\operatorname{Log} x - x^2 + \tfrac{1}{2}\operatorname{Log}(2\pi))$$
$$\left. + \tfrac{1}{3}\operatorname{Log} x\right\}, \tag{27.14}$$

by Stirling's formula (Entry 23 in Chapter 7).

On the other hand, by (27.8) and Entry 27(b),

$$\lim_{x\to\infty} \left\{2\sum_{k=1}^{x} k \operatorname{Log} k - x^2\operatorname{Log} x + \tfrac{1}{2}x^2 - x\operatorname{Log} x - \tfrac{1}{12}\operatorname{Log} x - \tfrac{1}{12}\right\}$$
$$= \frac{1}{6}\left\{\operatorname{Log}(2\pi) - \frac{\Gamma'(2)}{\Gamma(2)}\right\} + \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\operatorname{Log} k}{k^2}$$
$$= \tfrac{1}{6}\{\operatorname{Log}(2\pi) + \gamma - 1\} + \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\operatorname{Log} k}{k^2} \tag{27.15}$$

by (4.2) in Chapter 8. If we employ (27.15) in (27.14), we deduce that

$$L = \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\operatorname{Log} k}{k^2},$$

which completes the proof.

Example (ii). *We have*

$$\lim_{x\to\infty} e^{x^3/9 - x/12}\prod_{k=1}^{x}\left(\frac{k}{x}\right)^{k^2} = e^{\zeta(3)/(4\pi^2)}. \tag{27.16}$$

Proof. The logarithm of the left side of (27.16) is

$$L \equiv \lim_{x\to\infty} \left\{ \sum_{k=1}^{x} k^2(\text{Log}\, k - \text{Log}\, x) + \frac{x^3}{9} - \frac{x}{12} \right\}$$
$$= \lim_{x\to\infty} \left\{ \varphi_2(x) - \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} \right) \text{Log}\, x + \frac{x^3}{9} - \frac{x}{12} \right\}. \qquad (27.17)$$

On the other hand, by (27.7) and the Corollary above,

$$\lim_{x\to\infty} \left\{ \varphi_2(x) - \frac{x^3\,\text{Log}\, x}{3} + \frac{x^3}{9} - \frac{x^2\,\text{Log}\, x}{2} - \frac{x\,\text{Log}\, x}{6} - \frac{x}{12} \right\} = \frac{\zeta(3)}{4\pi^2}. \qquad (27.18)$$

Comparing (27.17) and (27.18), we readily deduce (27.16).

For each positive integer r, define

$$f(r, x) = \frac{H_r B_{r+1}(x+1)}{r+1} - \sum_{1 \le k \le (r+1)/2} \frac{B_{2k} r!\, H_{2k-1} x^{r-2k+1}}{(2k)!\,(r+1-2k)!}, \qquad (28.1)$$

where H_n is defined in (3.1) and $B_n(x)$ denotes the nth Bernoulli polynomial, $0 \le n < \infty$.

Entry 28(a). *For* $r \ge 2$,

$$f(r, x) = \frac{B_{r+1}(x+1) - B_{r+1}}{r(r+1)} + r \int_0^x f(r-1, t)\, dt. \qquad (28.2)$$

Proof. Since $B_{r+1}'(x) = (r+1)B_r(x)$, a direct calculation with the use of (28.1) yields

$$r \int_0^x f(r-1, t)\, dt = \frac{H_{r-1} B_{r+1}(x+1)}{r+1} - \frac{H_{r-1} B_{r+1}}{r+1}$$
$$- \sum_{1 \le k \le r/2} \frac{B_{2k} r!\, H_{2k-1} x^{r-2k+1}}{(2k)!\,(r+1-2k)!}$$
$$= f(r, x) - \frac{B_{r+1}(x+1)}{r(r+1)} - \frac{H_{r-1} B_{r+1}}{r+1} + \frac{H_r B_{r+1}}{r+1},$$

from which (28.2) immediately follows.

Ramanujan next studies an analytic extension of $\varphi_r(x)$ for all real values of x and any positive integer r. He does not give us his definition, but there exists considerable motivation for defining

$$\varphi_r(x) = \zeta'(-r, x+1) - \zeta'(-r), \qquad (28.3)$$

where $\zeta(s)$ denotes the Riemann zeta-function and $\zeta(s, a)$ denotes the Hurwitz zeta-function. Normally, the definition of $\zeta(s, a)$ requires the restriction

$0 < a \leq 1$. But as in Chapter 8, we shall remove this stipulation so that in the sequel a denotes any real number. In fact, we could allow a to be complex, but since Ramanujan evidently considered only real values of a, we shall do likewise.

First, note that if Re $s > 1$,

$$\zeta'(s, x+1) - \zeta'(s, x) = x^{-s} \operatorname{Log} x. \tag{28.4}$$

By analytic continuation, (28.4) is valid for all complex numbers s. Putting $s = -r$, we find that $\varphi_r(x) - \varphi_r(x-1) = x^r \operatorname{Log} x$ for any real number x. Since $\varphi_r(0) = 0$, we see that (28.3) is compatible with (27.1).

Secondly, (28.3) is similar to definitions of other analogues of $\operatorname{Log}\Gamma(x+1)$ studied in Chapter 8, and if $r = 0$, (28.3) reduces to a formula of Lerch for $\operatorname{Log}\Gamma(x+1)$. (See (18.12) in Chapter 8.)

Thirdly, if x and r are positive integers,

$$\sum_{k=1}^{x} k^r = \frac{B_{r+1}(x+1) - B_{r+1}}{r+1}, \tag{28.5}$$

where $B_n(x)$ denotes the nth Bernoulli polynomial and B_n denotes the nth Bernoulli number, $0 \leq n < \infty$. By Hurwitz's formula (17.15) in Chapter 8 and the Fourier expansion for $B_k(a)$,

$$\zeta(1-k, a) = -\frac{B_k(a)}{k}, \qquad k \geq 2, \quad 0 < a \leq 1. \tag{28.6}$$

Thus, we find that, for $-1 < x \leq 0$ and $r \geq 1$,

$$\frac{B_{r+1}(x+1) - B_{r+1}}{r+1} = -\zeta(-r, x+1) + \zeta(-r, 1). \tag{28.7}$$

If we formally differentiate (28.5) and (28.7) with respect to r and ignore the different restrictions on x, we formally deduce (28.3).

Entry 28(b). *If* $|x| < 1$ *and* r *is any positive integer, then*

$$\begin{aligned}\varphi_r(x) &= \frac{H_r}{r+1}\{B_{r+1}(x+1) - B_{r+1}\} \\ &= \frac{\gamma x^{r+1}}{r+1} - \sum_{1 \leq k \leq (r+1)/2} \frac{r!\, B_{2k} H_{2k-1} x^{r+1-2k}}{(2k)!\,(r+1-2k)!} \\ &\quad - \sum_{k=0}^{r-1} \binom{r}{k} C_k x^{r-k} + \sum_{k=2}^{\infty} \frac{(-1)^k r!\,(k-1)!\,\zeta(k) x^{r+k}}{(r+k)!},\end{aligned} \tag{28.8}$$

where H_n *is defined by* (3.1), γ *denotes Euler's constant, and* $C_k = -\zeta'(-k)$, $k \geq 0$.

The theory of a certain analytic extension of (28.3) has been extensively developed by Bendersky [1]. In fact, for $|x| < 1$, Bendersky [1, p. 279] defines his analytic extension by (28.8), except that the first sum on the right side of (28.8) does not appear in his definition. Büsing [1] has further developed Bendersky's work and has removed some deficiencies in Bendersky's definition of the constants L_k, which are closely related to C_k here.

Proof. For $\operatorname{Re} s > 1$ and $|x| < 1$,

$$\begin{aligned}
\zeta'(s, x+1) &= -\sum_{k=1}^{\infty} \frac{\operatorname{Log}(k+x)}{(k+x)^s} \\
&= -\sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{j=0}^{\infty} \binom{-s}{j}\left(\frac{x}{k}\right)^j \operatorname{Log}(k+x) \\
&= -\sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{j=0}^{\infty} \binom{-s}{j}\left(\frac{x}{k}\right)^j \left\{\operatorname{Log} k + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\left(\frac{x}{k}\right)^m\right\} \\
&= \sum_{j=0}^{\infty} \binom{-s}{j} \zeta'(s+j)x^j + \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{n=1}^{\infty} \left(\frac{x}{k}\right)^n \sum_{j=0}^{n-1} \binom{-s}{j} \frac{(-1)^{n+j}}{n-j} \\
&= \sum_{j=0}^{\infty} \binom{-s}{j} \zeta'(s+j)x^j + \sum_{n=1}^{\infty} \zeta(s+n)x^n \sum_{j=0}^{n-1} \binom{-s}{j} \frac{(-1)^{n+j}}{n-j} \\
&= S_1(s) + S_2(s) + S_3(s) + S_4(s) + S_5(s) + S_6(s),
\end{aligned} \tag{28.9}$$

where

$$S_1(s) = \sum_{j=0}^{r} \binom{-s}{j} \zeta'(s+j)x^j, \qquad S_2(s) = \binom{-s}{r+1} \zeta'(s+r+1)x^{r+1},$$

$$S_3(s) = \sum_{j=r+2}^{\infty} \binom{-s}{j} \zeta'(s+j)x^j,$$

$$S_4(s) = \sum_{n=1}^{r} \zeta(s+n)x^n \sum_{j=0}^{n-1} \binom{-s}{j} \frac{(-1)^{n+j}}{n-j},$$

$$S_5(s) = \zeta(s+r+1)x^{r+1} \sum_{j=0}^{r} \binom{-s}{j} \frac{(-1)^{r+1-j}}{r+1-j},$$

and

$$S_6(s) = \sum_{n=r+2}^{\infty} \zeta(s+n)x^n \sum_{j=0}^{n-1} \binom{-s}{j} \frac{(-1)^{n+j}}{n-j}.$$

By analytic continuation, the far right side of (28.9) represents $\zeta'(s, x+1)$ for all s.

We now evaluate

$$S_1(-r),\ S_3(-r),\ S_4(-r),\ S_6(-r), \quad \text{and} \quad \lim_{s \to -r}\{S_2(s) + S_5(s)\}.$$

First,

$$S_1(-r) = \sum_{j=0}^{r} \binom{r}{j} \zeta'(j-r)x^j = -\sum_{k=0}^{r} \binom{r}{k} C_k x^{r-k}, \tag{28.10}$$

since $\zeta'(-k) = -C_k$, $0 \le k \le r$, by (27.7).

Secondly, since $\binom{r}{j} = 0, j \geq r+2,$

$$S_3(-r) = 0. \tag{28.11}$$

Using (28.6) and the fact that $\zeta(0) = -\frac{1}{2}$, we find that

$$S_4(-r) = \sum_{n=1}^{r} \frac{B_{r-n+1}(1)}{r-n+1} x^n \sum_{j=0}^{n-1} \binom{r}{j} \frac{(-1)^{r+j+1}}{n-j}.$$

Employing the notation (I7) and the formula

$$\sum_{k=0}^{m-1} \frac{(a)_k}{k!\,(m-k)} = \frac{(a)_m}{m!} \sum_{k=0}^{m-1} \frac{1}{a+k}, \tag{28.12}$$

found in Hansen's tables [1, p. 126], we find that

$$\begin{aligned}
S_4(-r) &= \sum_{n=1}^{r} \frac{B_{r+1-n}(1)(-1)^{r+1}(-r)_n x^n}{(r+1-n)n!} \sum_{k=0}^{n-1} \frac{1}{-r+k} \\
&= \frac{1}{r+1} \sum_{n=1}^{r} \binom{r+1}{n} B_{r+1-n}(1) x^n \sum_{k=0}^{n-1} \frac{1}{r-k} \\
&= \frac{1}{r+1} \sum_{n=1}^{r} \binom{r+1}{n} B_{r+1-n}(1) x^n (H_r - H_{r-n}) \\
&= \frac{H_r}{r+1} \sum_{n=1}^{r} \binom{r+1}{n} B_{r+1-n}(1) x^n \\
&\quad - \frac{1}{r+1} \sum_{k=1}^{r} \binom{r+1}{k} B_k(1) H_{k-1} x^{r+1-k} \\
&= \frac{H_r}{r+1} \{B_{r+1}(x+1) - B_{r+1} - x^{r+1}\} \\
&\quad - \sum_{1 \leq k \leq (r+1)/2} \frac{r!\, B_{2k} H_{2k-1} x^{r+1-2k}}{(2k)!\,(r+1-2k)!},
\end{aligned} \tag{28.13}$$

where we have used a familiar formula for $B_{r+1}(x+h)$ (Abramowitz and Stegun [1, formula 23.17, p. 804]).

Letting $n = k + r$ and $j = r - m$, we find that

$$S_6(-r) = \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^{r+k} \sum_{m=0}^{r} \binom{r}{m} \frac{(-1)^m}{m+k}.$$

Now if k is a positive integer, from Gould's tables [8, formula 1.41, p. 6],

$$\sum_{m=0}^{r} \binom{r}{m} \frac{(-1)^m}{m+k} = \frac{r!\,(k-1)!}{(r+k)!}. \tag{28.14}$$

Thus,

$$S_6(-r) = \sum_{k=2}^{\infty} \frac{(-1)^k r!\,(k-1)!\,\zeta(k) x^{r+k}}{(r+k)!}. \tag{28.15}$$

We lastly examine $S_2(s) + S_5(s)$ as s tends to $-r$. Letting

$$f(s) = \sum_{j=0}^{r} \binom{-s}{j} \frac{(-1)^{r+j+1}}{r+1-j},$$

we have

$$\begin{aligned} L &\equiv \lim_{s \to -r} \{S_2(s) + S_5(s)\} \\ &= x^{r+1} \lim_{s \to -r} \left\{ \binom{-s}{r+1} \zeta'(s+r+1) + f(s)\zeta(s+r+1) \right\}. \end{aligned} \tag{28.16}$$

Replacing j by $r-k$ and using (28.14), we find that

$$f(-r) = -\sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^k}{k+1} = -\frac{1}{r+1}. \tag{28.17}$$

Since

$$f'(s) = \sum_{j=1}^{r} \frac{(-1)^{r+1}}{(r+1-j)j!} \sum_{k=0}^{j-1} \prod_{\substack{m=0 \\ m \neq k}}^{j-1} (s+m),$$

we see that

$$\begin{aligned} f'(-r) &= \sum_{j=1}^{r} \frac{(-1)^{r+j}}{r+1-j} \binom{r}{j} \sum_{k=0}^{j-1} \frac{1}{r-k} \\ &= \sum_{j=1}^{r} \frac{(-1)^{r+j}}{r+1-j} \binom{r}{j} (H_r - H_{r-j}) \\ &= T_1 + T_2, \end{aligned} \tag{28.18}$$

say. We first calculate T_1. We have

$$\begin{aligned} T_1 &= \frac{(-1)^r H_r}{r+1} \sum_{j=1}^{r} (-1)^j \binom{r+1}{j} \\ &= \frac{(-1)^r H_r}{r+1} \{-1 - (-1)^{r+1}\} \\ &= \frac{H_r}{r+1} \{1 - (-1)^r\}. \end{aligned} \tag{28.19}$$

Inverting the order of summation and, in the third equality below, using a well-known evaluation (Gradshteyn and Ryzhik [1, p. 3]), we find that

$$
\begin{aligned}
T_2 &= \sum_{j=1}^{r-1} \frac{(-1)^{r+j+1}}{r+1-j}\binom{r}{j}\sum_{k=1}^{r-j}\frac{1}{k} \\
&= \frac{(-1)^{r+1}}{r+1}\sum_{k=1}^{r-1}\frac{1}{k}\sum_{j=1}^{r-k}(-1)^j\binom{r+1}{j} \\
&= \frac{(-1)^{r+1}}{r+1}\sum_{k=1}^{r-1}\frac{1}{k}\left\{(-1)^{r+k}\binom{r}{r-k}-1\right\} \\
&= \frac{(-1)^{r+1}}{r+1}\sum_{k=1}^{r-1}\frac{(-1)^k}{r-k}\binom{r}{k}+\frac{(-1)^r}{r+1}H_{r-1} \\
&= \frac{(-1)^{r+1}(-r)_r}{(r+1)r!}\sum_{k=0}^{r-1}\frac{1}{-r+k}+\frac{(-1)^r}{r+1}H_r \\
&= \frac{1}{r+1}\{1+(-1)^r H_r\},
\end{aligned} \tag{28.20}
$$

where in the penultimate equality we used (28.12) again. Employing (28.19) and (28.20) in (28.18), we deduce that

$$
f'(-r) = \frac{2H_r}{r+1}. \tag{28.21}
$$

We now return to (28.16) and expand each function on the right side about $z=0$, where $z=s+r$. The expansion for $1/\Gamma(-z)$ and the coefficient of z in the Maclaurin series of $\Gamma(r-z+1)$ can be found in the tables of Gradshteyn and Ryzhik [1, pp. 936, 945]. For the first two terms in the Laurent expansion of $\zeta(z+1)$, see Entry 13 of Chapter 7. Hence, using also (28.17) and (28.21), we find that

$$
\begin{aligned}
&\frac{\Gamma(-s+1)}{(r+1)!\,\Gamma(-s-r)}\zeta'(s+r+1)+f(s)\zeta(s+r+1) \\
&\quad= \frac{1}{(r+1)!}\{-z+\gamma z^2+\cdots\}\{r!+(\gamma-H_r)r!\,z+\cdots\}\left\{-\frac{1}{z^2}+c_1+\cdots\right\} \\
&\qquad+\left\{-\frac{1}{r+1}+\frac{2H_r}{r+1}z+\cdots\right\}\left\{\frac{1}{z}+\gamma+\cdots\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L &= x^{r+1}\left(\frac{1}{r+1}(\gamma-H_r)-\frac{\gamma}{r+1}-\frac{\gamma}{r+1}+\frac{2H_r}{r+1}\right) \\
&= \frac{x^{r+1}}{r+1}(H_r-\gamma).
\end{aligned} \tag{28.22}
$$

If we now utilize (28.10), (28.11), (28.13), (28.15), and (28.22) in (28.9) with $s=-r$ and recall the definition of $\varphi_r(x)$ in (28.3), we readily deduce (28.8).

Entry 29. *Let $\varphi_r(x)$ be defined by (28.3), let C_r be defined by (27.7), let $-1 < x \le 0$, and let n and r be natural numbers. Then*

$$\varphi_r(x) - n^r \sum_{k=0}^{n-1} \varphi_r\left(\frac{x-k}{n}\right) = \frac{B_{r+1}(x+1)}{r+1} \operatorname{Log} n + (1 - n^{r+1})C_r.$$

Proof. Let Re $s > 1$ and replace k by $n-k$ below to get

$$\begin{aligned}\zeta'(s, x+1) - n^r \sum_{k=0}^{n-1} \zeta'\left(s, \frac{x-k}{n} + 1\right) \\ &= \zeta'(s, x+1) + n^r \sum_{k=1}^{n} \sum_{j=0}^{\infty} \frac{\operatorname{Log}\{(jn+k+x)/n\}}{\{(jn+k+x)/n\}^s} \\ &= \zeta'(s, x+1) + n^{r+s} \sum_{m=1}^{\infty} \frac{\operatorname{Log}\{(m+x)/n\}}{(m+x)^s} \\ &= (1 - n^{r+s})\zeta'(s, x+1) - n^{r+s} \operatorname{Log} n\, \zeta(s, x+1).\end{aligned}$$

By analytic continuation, the extremal sides of the equalities above are equal for all complex s. Now let $s = -r$ and use (28.3) and (27.7) to obtain

$$\varphi_r(x) - n^r \sum_{k=0}^{n-1} \varphi_r\left(\frac{x-k}{n}\right) = -\zeta(-r, x+1) \operatorname{Log} n + (1 - n^{r+1})C_r.$$

Noting that $-1 < x \le 0$, we may employ (28.6) to complete the proof.

Corollary 1. *If n and r are any positive integers, then*

$$\sum_{k=1}^{n-1} \varphi_r\left(-\frac{k}{n}\right) = -\frac{B_{r+1} \operatorname{Log} n}{(r+1)n^r} + (n - n^{-r})C_r.$$

Proof. Putting $x = 0$ in Entry 29 and recalling that $\varphi_r(0) = 0$, we easily deduce the desired equality.

Corollary 2. *If r is any positive integer, then*

$$\varphi_r(-\tfrac{1}{2}) = -\frac{B_{r+1} \operatorname{Log} 2}{(r+1)2^r} + (2 - 2^{-r})C_r.$$

Proof. Set $n = 2$ in Corollary 1.

Entry 30. *Let $0 < x < 1$. If r is positive and even, then*

(i) $$\varphi_r(x-1) + \varphi_r(-x) = 2C_r + \frac{r!}{(2\pi)^r} \cos\left(\frac{\pi r}{2}\right) \sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{k^{r+1}};$$

if r is positive and odd, then

(ii) $$\varphi_r(x-1) - \varphi_r(-x) = \frac{r!}{(2\pi)^r} \sin\left(\frac{\pi r}{2}\right) \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k^{r+1}}.$$

Proof. Recall Hurwitz's formula given by (17.15) in Chapter 8. Differentiating with respect to s, we find that

$$\zeta'(s,x) = -2\Gamma'(1-s)\left\{\sin\left(\frac{\pi s}{2}\right)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{(2\pi k)^{1-s}} + \cos\left(\frac{\pi s}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{(2\pi k)^{1-s}}\right\}$$

$$+\pi\Gamma(1-s)\left\{\cos\left(\frac{\pi s}{2}\right)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{(2\pi k)^{1-s}} - \sin\left(\frac{\pi s}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{(2\pi k)^{1-s}}\right\}$$

$$+2\Gamma(1-s)\left\{\sin\left(\frac{\pi s}{2}\right)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)\operatorname{Log}(2\pi k)}{(2\pi k)^{1-s}}\right.$$

$$\left.+\cos\left(\frac{\pi s}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)\operatorname{Log}(2\pi k)}{(2\pi k)^{1-s}}\right\}. \tag{30.1}$$

Thus, if $0 < x < 1$ and r is even,

$$\varphi_r(x-1) + \varphi_r(-x) = \zeta'(-r, x) + \zeta'(-r, 1-x) + 2C_r$$
$$= 2\pi\Gamma(1+r)\cos\left(\frac{\pi r}{2}\right)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{(2\pi k)^{r+1}} + 2C_r,$$

which completes the proof of (i). The proof of (ii) is analogous.

Ramanujan remarks that "More general theorems true for all values of r can be got ..." (p. 114). Indeed, (28.3) can be used to define $\varphi_r(x)$ for any real number r, $r \neq -1$. Thus, (30.1) can then be employed to obtain generalizations of (i) and (ii) for $r > -1$.

Entry 31(a). *Suppose that* $\varphi_1(x)$ *is defined by* (28.3), *and define*

$$\psi(x) = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{\sin^{2k+1}(\pi x)}{2^{2k}(2k+1)^2}. \tag{31.1}$$

Then if $0 < x \leq \frac{1}{2}$,

$$\psi(x) = \pi\{\varphi_1(x-1) - \varphi_1(-x)\} + \pi x \operatorname{Log}(2\sin(\pi x)).$$

Proof. By Entry 30(ii), for $0 < x < 1$,

$$\pi\{\varphi_1(x-1) - \varphi_1(-x)\} + \pi x \operatorname{Log}(2\sin(\pi x))$$
$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^2} + \pi x \operatorname{Log}(2\sin(\pi x)).$$

But by Entry 16, with $|x| \leq \frac{1}{2}$,

$$\psi(x) = \pi x \operatorname{Log}|2\sin(\pi x)| + \frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^2}.$$

The desired result now follows.

Entry 31(b). *Let* $\psi(x)$ *be defined by* (31.1) *and let* h_k *be defined by* (8.2). *Then we have*

(i)
$$\psi(x) = \sum_{k=0}^{\infty} \frac{(-1)^k h_{k+1}}{2k+1} \tan^{2k+1}(\pi x), \qquad |x| \le \tfrac{1}{4},$$

(ii)
$$\psi(x) + \psi(\tfrac{1}{2} - x) = \frac{\pi}{2} \operatorname{Log}(2\cos(\pi x)) + \sum_{k=0}^{\infty} \frac{(-1)^k \tan^{2k+1}(\pi x)}{(2k+1)^2}, \qquad 0 \le x < \tfrac{1}{2},$$

(iii)
$$\psi(\tfrac{1}{2} - x) + \tfrac{1}{2}\psi(2x) - \psi(x) = \frac{\pi}{2} \operatorname{Log}(2\cos(\pi x)), \qquad 0 \le x \le \tfrac{1}{4},$$

and

(iv)
$$\psi(\tfrac{1}{2} - x) + \psi(\tfrac{1}{2} + x) = \pi(1 - 2x) \operatorname{Log}|2\cos(\pi x)| + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sin(2\pi k x)}{k^2}, \qquad 0 \le x \le 1.$$

Proof of (i). By the Cauchy multiplication of the Maclaurin series of $(\tan^{-1} t)/t$ and $1/(1+t^2)$, or by Proposition 15(i), we find that, for $|t| < 1$,

$$\frac{\tan^{-1} t}{t(1+t^2)} = \sum_{k=0}^{\infty} (-1)^k h_{k+1} t^{2k}.$$

Thus, for $|x| \le \frac{1}{4}$, we find that the right side of (i) is equal to

$$\int_0^{\tan(\pi x)} \frac{\tan^{-1} t}{t(1+t^2)}\, dt = \int_0^{\pi x} u \cot u\, du$$
$$= \pi x \operatorname{Log}|2\sin(\pi x)| + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k^2}. \qquad (31.2)$$

In the penultimate step we made the substitution $u = \tan^{-1} t$, and in the last step we applied Entry 13 with $n = 1$ and used (13.1) and (13.3). Using Entry 16 on the far right side of (31.2) along with the definition (31.1) of $\psi(x)$, we complete the proof.

Proof of (ii). From the definition (31.1) of $\psi(x)$, we find that, for $0 \le x < \frac{1}{2}$,

$$\psi(x) + \psi(\tfrac{1}{2} - x) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\sin^{2k+1}(\pi x) + \cos^{2k+1}(\pi x)}{2^{2k}(2k+1)^2}.$$

Applying Entry 19, we finish the proof.

Proof of (iii). By Entry 18, for $0 \le x \le \frac{1}{4}$,

$$\psi(\tfrac{1}{2} - x) - \psi(x) = \frac{\pi}{2} \operatorname{Log}(2\cos(\pi x)) - \frac{1}{2} \sum_{k=0}^{\infty} \binom{2k}{k} \frac{\sin^{2k+1}(2\pi x)}{2^{2k}(2k+1)^2}.$$

Using the definition (31.1), we complete the proof.

Proof of (iv). Using the definition (31.1) and (18.2), we find that, for $0 \le x \le 1$,

$$\psi(\tfrac{1}{2}-x)+\psi(\tfrac{1}{2}+x) = 2\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\cos^{2k+1}(\pi x)}{2^{2k}(2k+1)^2}$$
$$= 2\left(\frac{\pi}{2}-\pi x\right)\operatorname{Log}|2\cos(\pi x)|$$
$$+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin(2\pi k x)}{k^2},$$

which completes the proof.

Part (iv) is not what is claimed by Ramanujan, p. 115. Ramanujan asserts that

$$\psi(\tfrac{1}{2}-x)+\psi(\tfrac{1}{2}+x) = \pi\operatorname{Log}(2\cos(\pi x)).$$

Evidently, Ramanujan applied Entry 16 twice, with x replaced by $\frac{1}{2}\pi - \pi x$ and with x replaced by $\frac{1}{2}\pi + \pi x$. But the intersection of the domains for which these two equalities are valid is only the origin. The infinite series on the right side of (iv) cannot be evaluated in terms of elementary functions. In fact, from Gradshteyn and Ryzhik's tables [1, formula 1.441; 4, p. 38], it is easily seen that, for $0 \le x \le 1$,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin(2\pi k x)}{k^2} = 2\pi\int_0^x \operatorname{Log}|2\cos(\pi t)|\,dt.$$

Examples. *We have*

(i) $$\psi(\tfrac{1}{2}) = \frac{\pi}{2}\operatorname{Log} 2,$$

(ii) $$\psi(\tfrac{1}{4}) = \tfrac{1}{2}G + \frac{\pi}{8}\operatorname{Log} 2,$$

(iii) $$\psi(\tfrac{1}{3}) = \frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2} - \frac{\pi^2}{9\sqrt{3}} + \frac{\pi}{6}\operatorname{Log} 3,$$

(iv) $$\psi(\tfrac{1}{6}) = \frac{3\sqrt{3}}{4}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2} - \frac{\pi^2}{6\sqrt{3}},$$

and

(v) $$2\psi(x) - \tfrac{1}{2}\psi(2x) = \sum_{k=0}^{\infty}\frac{(-1)^k\tan^{2k+1}(\pi x)}{(2k+1)^2}, \qquad 0 \le x \le \tfrac{1}{4}.$$

Proof. Parts (i)–(iv) are merely restatements of Examples (i)–(iv) in Section 16. Part (v) arises from Entry 31(b) by subtracting (iii) from (ii).

In Ramanujan's version, p. 115, of part (ii) above, read $2\psi(\frac{1}{4})$ for $\psi(\frac{1}{4})$. Entry 32 below should be compared with Entry 17.

Entry 32. *For* $|x| \leq \pi/4$, *we have*

$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2 \sin^{2k+1}(2x)}{(2k)!\,(2k+1)^2} = 2 \sum_{k=0}^{\infty} \frac{(-1)^k \tan^{2k+1} x}{(2k+1)^2}. \tag{32.1}$$

Proof. Using Proposition 15(ii) and then making the successive substitutions $t = \sin(2u)$ and $u = \tan^{-1} v$, we find that the left side of (32.1) is equal to, for $|x| \leq \pi/4$,

$$\begin{aligned}
\int_0^{\sin(2x)} \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2 t^{2k}}{(2k+1)!}\,dt &= \int_0^{\sin(2x)} \frac{\sin^{-1} t}{t\sqrt{1-t^2}}\,dt \\
&= 4 \int_0^x u \csc(2u)\,du = 2 \int_0^{\tan x} \frac{\tan^{-1} v}{v}\,dv \\
&= 2 \int_0^{\tan x} \sum_{k=0}^{\infty} \frac{(-1)^k v^{2k}}{2k+1}\,dv = 2 \sum_{k=0}^{\infty} \frac{(-1)^k \tan^{2k+1} x}{(2k+1)^2}.
\end{aligned}$$

Entry 32 also falls in the realm of generalized hypergeometric series. In Whipple's quadratic transformation of a well poised ${}_3F_2$ (Erdélyi [1, p. 190])

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, b, c \\ a+1-b, a+1-c \end{matrix}; t\right] \\
&\qquad = (1-t)^{-a}\,{}_3F_2\left[\begin{matrix} a+1-b-c, \frac{1}{2}a, \frac{1}{2}(a+1) \\ a+1-b, a+1-c \end{matrix}; \frac{-4t}{(1-t)^2}\right],
\end{aligned}$$

let $a = 1$, $b = c = \frac{1}{2}$, and $t = -\tan^2 x$. After some simplification, we readily deduce (32.1).

Corollary (i). *For* $|x| \leq 1$, *we have*

$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(2k)!\,(2k+1)^2} \left(\frac{4x}{(1+x)^2}\right)^k = (1+x) \sum_{k=0}^{\infty} \frac{(-x)^k}{(2k+1)^2}. \tag{32.2}$$

Proof. Replace $\tan x$ by $\sqrt{u}$ in Entry 32. Noting that

$$\sin(2x) = (2 \tan x)/(1 + \tan^2 x) = 2\sqrt{u}/(1+u),$$

we readily deduce (32.2) with x replaced by u.

Ramanujan, p. 115, has a slight misprint in the third summand on the left side of (32.2).

Corollary (ii). *If* $|x| \leq \pi/4$, *then*

$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}(k!)^2 \tan^{2k+1}(2x)}{(2k)!\,(2k+1)^2} = 2 \sum_{k=0}^{\infty} \frac{\tan^{2k+1} x}{(2k+1)^2}.$$

Proof. In Corollary (i), replace x by $-\tan^2 u$. Noting that $\tan^2(2u) = 4\tan^2 u/(1-\tan^2 u)^2 = -4x/(1+x)^2$, we easily achieve the proposed identity with x replaced by u.

Examples. *We have*

(i)
$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(2k)!\,(2k+1)^2} = 2G,$$

(ii)
$$\sum_{k=0}^{\infty} \frac{3^k(k!)^2}{(2k)!\,(2k+1)^2} = -\frac{\pi}{3\sqrt{3}} \operatorname{Log} 3 - \frac{10\pi^2}{27} + 5 \sum_{k=0}^{\infty} \frac{1}{(3k+1)^2},$$

(iii)
$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!\,(2k+1)^2} = \frac{8}{3}G - \frac{\pi}{3} \operatorname{Log}(2+\sqrt{3}),$$

(iv)
$$\sum_{k=0}^{\infty} \frac{2^k(k!)^2}{(2k)!\,(2k+1)^2} = -\frac{\pi}{2\sqrt{2}} \operatorname{Log}(1+\sqrt{2}) - \frac{\pi^2}{4\sqrt{2}} + 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)^2},$$

(v)
$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(2k)!\,(2k+1)^2}\left(1 - \frac{3}{4^{k+1}}\right) = \frac{\pi}{4} \operatorname{Log}(2+\sqrt{3}),$$

(vi)
$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}(k!)^2}{(2k)!\,(2k+1)^2} = \frac{\pi^2}{8} - \tfrac{1}{2} \operatorname{Log}^2(1+\sqrt{2}),$$

and

(vii)
$$\sum_{k=0}^{\infty} \frac{(-1)^k(k!)^2}{(2k)!\,(2k+1)^2} = \frac{\pi^2}{6} - 3 \operatorname{Log}^2\left(\frac{\sqrt{5}+1}{2}\right).$$

Proof. Example (i) follows from putting $x = 1$ in Corollary (i) or $x = \pi/4$ in Entry 32.

Putting $x = \frac{1}{3}$ in Corollary (i) yields

$$\sum_{k=0}^{\infty} \frac{3^k(k!)^2}{(2k)!\,(2k+1)^2} = \frac{4}{3} \sum_{k=0}^{\infty} \frac{(-\frac{1}{3})^k}{(2k+1)^2}.$$

Employing Example 17(i), we deduce part (ii).

Thirdly, put $x = (2-\sqrt{3})^2$ in Corollary (i). Then $4x/(1+x)^2 = \frac{1}{4}$, and so we get

$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!\,(2k+1)^2} = 4 \sum_{k=0}^{\infty} \frac{(-1)^k(2-\sqrt{3})^{2k+1}}{(2k+1)^2}.$$

Applying Example 17(iii), we readily deduce (iii) above.

Next, put $x = (\sqrt{2} - 1)^2$ in Corollary (i). Then $4x/(1+x)^2 = \frac{1}{2}$, and thus

$$\sum_{k=0}^{\infty} \frac{2^k (k!)^2}{(2k)!\,(2k+1)^2} = 2\sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{2}-1)^{2k+1}}{(2k+1)^2}.$$

Appealing to Example 17(ii), we readily find (iv).

Fifthly, multiply the formula of part (iii) by $-\frac{3}{4}$ and add it to the formula of part (i). This yields part (v).

Next, let $\tan x = \tan(\pi/8) = \sqrt{2} - 1$. Then $\tan(2x) = 1$, and so Corollary (ii) yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} (k!)^2}{(2k)!\,(2k+1)^2} = 2 \sum_{k=0}^{\infty} \frac{(\sqrt{2}-1)^{2k+1}}{(2k+1)^2}.$$

Now use Example 6(iv) to deduce formula (vi).

Lastly, let $\tan x = \sqrt{5} - 2$ in Corollary (ii). Then $\tan(2x) = \frac{1}{2}$. Using Example 6(vi), we readily achieve (vii) above.

Example (i) is originally due to Nielsen [4, p. 166] and is reminiscent of the formula (0.1) for $\zeta(3)$ that was used by Apéry [1] to prove the irrationality of $\zeta(3)$. An interesting, animated account of Apéry's proof has been written by van der Poorten [1]. Mendès France [1] has also described the lecture in which Apéry announced his achievement. In fact, formula (0.1) appears to be originally due to Hjortnaes [1] in 1953. Other proofs have been given by D. Hawkins (personal communication, January, 1977), Ayoub (personal communication, 1974), and van der Poorten [2]. Cohen [2] and Leshchiner [1] have established different formulas for $\zeta(n)$, $2 \le n < \infty$, for which (0.1) is a special case. Other results in the spirit of Examples (i)–(vii) and (0.1) may be found in Comtet's book [1, p. 89] and in the papers of Clausen [2], Ruscheweyh [1], van der Poorten [2], [3], Zucker [1], and Gosper [1]. It is interesting that the formula for G in Example (i) was discovered almost a half century before the formula (0.1) for $\zeta(3)$.

Entry 33. *If n is a positive integer, then*

(i) $$\int_0^{\pi/2} x \cos^n x \sin(nx)\, dx = \frac{\pi}{2^{n+2}} H_n,$$

where H_n is defined in (3.1), *and*

(ii) $$\int_0^{\pi/2} \cos^n x \sin(nx)\, dx = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k}.$$

Proof. We shall prove only (i). Formula (ii) is slightly easier to establish, and a proof may be found in Fichtenholz's text [1, p. 136].

By Entry 5(i) and an integration by parts,

$$\int_0^{\pi/2} x \cos^n x \sin(nx)\, dx = \frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k} \int_0^{\pi/2} x \sin(2kx)\, dx$$
$$= \frac{\pi}{2^{n+2}} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k}$$
$$= \frac{\pi}{2^{n+2}} H_n,$$

by a well-known formula (Gradshteyn and Ryzhik [1, p. 4]).

The next two results are designated by Ramanujan as corollaries of Entry 33. However, we prefer to begin the proofs anew. On the surface, it appears that these two corollaries as written by Ramanujan are devoid of meaning. However, each can be assigned a mathematically precise meaning.

Ramanujan defines

$$\varphi(n) = \sum_{k=1}^{n} \frac{2^k - 1}{k}$$

and claims, p. 116, that $\varphi(n)$ "can be expanded in ascending powers of n in a convergent series the first two terms being $S_2 x/2 + S_3 x^2/8 + \&c.$" Here $S_k = \zeta(k)$, $k \geq 2$. We shall need to extend the definition of $\varphi(x)$ to all real values of x. Upon integrating both sides of

$$\sum_{k=1}^{x} t^{k-1} = \frac{t^x - 1}{t - 1}$$

over $1 \leq t \leq 2$, we find that

$$\varphi(x) = \int_1^2 \frac{t^x - 1}{t - 1}\, dt. \tag{33.1}$$

Thus, we shall define $\varphi(x)$ for every real number x by (33.1). By expanding $t^x - 1$ in a power series and inverting the order of summation and integration, we find that

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{a_k x^k}{k!},$$

where

$$a_k = \int_1^2 \frac{\operatorname{Log}^k t}{t - 1}\, dt, \qquad k \geq 1. \tag{33.2}$$

We now state a revised version of Ramanujan's first corollary.

Corollary 1. *If a_k is defined by* (33.2), *then $a_1 = \frac{1}{2}\zeta(2)$ and $a_2 = \frac{1}{4}\zeta(3)$.*

Despite the fact that a_k is a rational multiple of $\zeta(k+1)$ for $k = 1, 2$, it does not appear that this property persists for $k \geq 3$. (See Lewin's book [1, p. 199].)

Proof. Integrating by parts and using Example (i) in Section 6, we find that

$$\begin{aligned} a_1 &= \int_1^2 \frac{\operatorname{Log} t}{t-1}\, dt = \sum_{k=0}^{\infty} \int_1^2 \frac{\operatorname{Log} t}{t^{k+1}}\, dt \\ &= \tfrac{1}{2} \operatorname{Log}^2 2 + \sum_{k=1}^{\infty} \left\{ -\frac{t^{-k}}{k} \operatorname{Log} t \Big|_1^2 + \frac{1}{k} \int_1^2 \frac{dt}{t^{k+1}} \right\} \\ &= \tfrac{1}{2} \operatorname{Log}^2 2 - \operatorname{Log} 2 \sum_{k=1}^{\infty} \frac{1}{k2^k} - \operatorname{Li}_2(\tfrac{1}{2}) + \zeta(2) \\ &= -\tfrac{1}{2} \operatorname{Log}^2 2 - \frac{\pi^2}{12} + \tfrac{1}{2} \operatorname{Log}^2 2 + \zeta(2) = \tfrac{1}{2}\zeta(2), \end{aligned}$$

as claimed.

Next, integrating by parts twice and employing Example (i) in both Sections 6 and 7, we find that

$$\begin{aligned} a_2 &= \int_1^2 \frac{\operatorname{Log}^2 t}{t-1}\, dt = \sum_{k=0}^{\infty} \int_1^2 \frac{\operatorname{Log}^2 t}{t^{k+1}}\, dt \\ &= \tfrac{1}{3} \operatorname{Log}^3 2 - \operatorname{Log}^3 2 - 2 \operatorname{Log} 2 \operatorname{Li}_2(\tfrac{1}{2}) \\ &\quad - 2 \operatorname{Li}_3(\tfrac{1}{2}) + 2\zeta(3) = \tfrac{1}{4}\zeta(3), \end{aligned}$$

since $\chi_3(1) = \frac{7}{8}\zeta(3)$.

Corollary 2. *For each positive integer n,*

$$\varphi(-n) = -\sum_{k=n}^{\infty} \frac{1}{k2^k} - H_{n-1},$$

where H_n is defined by (3.1).

Proof. From (33.1),

$$\begin{aligned} \varphi(-n) &= \int_1^2 \frac{t^{-n} - 1}{t-1}\, dt = -\int_1^2 \sum_{k=0}^{n-1} t^{-k-1}\, dt \\ &= -\operatorname{Log} 2 + \sum_{k=1}^{n-1} \frac{1}{k2^k} - \sum_{k=1}^{n-1} \frac{1}{k} \\ &= -\sum_{k=n}^{\infty} \frac{1}{k2^k} - H_{n-1}. \end{aligned}$$

Ramanujan, p. 116, next seems to indicate that Corollary 2, perhaps in conjunction with Corollary 1, can be used to find the value of $\operatorname{Li}_k(\frac{1}{2})$, $k \geq 2$, where Li_k is defined by (6.1). The calculations in the proof of Corollary 1 make it clear that $\text{Li}_{k+1}(\frac{1}{2})$ arises in the calculation of a_k, $k \geq 1$. Since Corollary 2 is valid only when n is a positive integer, it does not appear that these last two facts can be utilized to determine $\text{Li}_k(\frac{1}{2})$.

Entry 34. *Let* $-\frac{1}{2} < x < 1$. *Then*

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{x}{1+x} \right)^{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} (k!)^2 h_{k+1} x^{k+1}}{(2k+1)!}, \tag{34.1}$$

where h_k *is defined in* (8.2).

Proof. Rearranging the double series below by absolute convergence, we find that, for $-\frac{1}{2} < x < 1$,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{x}{1+x} \right)^{k+1} &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)^2} \sum_{j=0}^{\infty} \binom{-k-1}{j} x^j \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2} \binom{n}{k} x^{n+1}. \end{aligned} \tag{34.2}$$

Comparing (34.1) and (34.2), we see that we must show that

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2} \binom{n}{k} = \frac{2^{2n}(n!)^2}{(2n+1)!} h_{n+1}, \qquad n \geq 0. \tag{34.3}$$

Since

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} t^{2k} = (1-t^2)^n,$$

we find after two integrations that

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2} \binom{n}{k} &= \int_0^1 \frac{dx}{x} \int_0^x (1-t^2)^n \, dt \\ &= \int_0^1 (1-t^2)^n \, dt \int_t^1 \frac{dx}{x} \\ &= -\int_0^1 (1-t^2)^n \operatorname{Log} t \, dt. \end{aligned} \tag{34.4}$$

Letting $B(x, y)$ denote the beta function and $\psi(x)$ the logarithmic derivative of the gamma function, we find that the far right side of (34.4) is equal to (Gradshteyn and Ryzhik [1, formulas 4.253, 1, p. 538; 8.363, 3, p. 944])

$$\begin{aligned} &-\tfrac{1}{4} B(\tfrac{1}{2}, n+1) \{ \psi(\tfrac{1}{2}) - \psi(n+1+\tfrac{1}{2}) \} \\ &\qquad = \frac{\sqrt{\pi}\, n!}{4\Gamma(n+\frac{3}{2})} \sum_{k=0}^{\infty} \left(\frac{1}{\frac{1}{2}+k} - \frac{1}{\frac{1}{2}+n+1+k} \right) \\ &\qquad = \frac{2^{2n}(n!)^2}{(2n+1)!} h_{n+1}, \end{aligned}$$

where we have used the Legendre duplication formula. This completes the proof of (34.3) and hence of (34.1) as well.

If we let x tend to $-\frac{1}{2}$ in (34.1), we obtain a formula for Catalan's constant that has been found in a different way by Fee [1].

In preparation for the last theorem in Chapter 9, define

$$K_r = \sum_{k=1}^{\infty} \frac{1}{k^r(k+1)^r} \quad \text{and} \quad A_n = (1+\cos(\pi n))\zeta(n),$$

where r is a positive integer and n is any integer. If $n = 1$, it is to be understood that $A_1 = 0$. Since $\zeta(-2k) = 0$ for each positive integer k, it follows that $A_n = 0$ if $n < 0$. Also, since $\zeta(0) = -\frac{1}{2}$, we have $A_0 = -1$.

Entry 35. *For each positive integer r,*

$$K_r = \sum_{k=0}^{\infty} (-1)^k \binom{r+k-1}{k} A_{r-k}. \tag{35.1}$$

By the remarks made above, the series on the right side of (35.1) terminates. Formula (35.1), or formulas easily equivalent to it, are well known. The first proof of (35.1) is apparently due to Glaisher [3] in 1913. Later proofs were found by Kesava Menon [2] and Djoković [1]. One can also find (35.1) in Hansen's tables [1, p. 116].

Examples. *We have*

(i) $$K_2 = \frac{\pi^2}{3} - 3,$$

(ii) $$K_3 = 10 - \pi^2,$$

(iii) $$K_4 = \frac{\pi^4}{45} + \frac{10\pi^2}{3} - 35,$$

and

(iv) $$K_5 = 126 - \frac{35\pi^2}{3} - \frac{\pi^4}{9}.$$

All of these examples are consequences of (35.1). Formulas (i)–(iii) are given explicitly by Hansen [1, pp. 40, 35, 31]. Greenstreet [1] found (i)–(iii) in 1907, six years before Glaisher [3] found (35.1). One can also find (i) in Bromwich's text [1, p. 226].

Ramanujan's Quarterly Reports

Introduction

In 1910, Ramanujan met with V. R. Aiyar, the founder of the Indian Mathematical Society, in the hope of securing clerical employment in a municipal office of Tirukoilur. With this meeting, word of Ramanujan's mathematical genius slowly began to spread amongst mathematicians in southeast India. Several people, including R. Ramachandra Rao, P. V. Seshu Aiyar, S. N. Aiyar, Sir Francis Spring, and Sir Gilbert Walker, took a kindly interest in Ramanujan through financial support, employment, and encouragement. In particular, on February 26, 1913, the English meteorologist Walker sent a letter to the registrar of the University of Madras, Francis Dewsbury, with the emphatic recommendation, "The University would be justified in enabling S. Ramanujan for a few years *at least* to spend the whole of his time on mathematics, without any anxiety as to his livelihood." The Board of Studies at the University of Madras agreed to this request, and its chairman, Professor B. Hanumantha Rao, wrote a letter to the Vice-Chancellor on March 25, 1913, with an exhortation that Ramanujan be awarded a scholarship of 75 rupees per month. Again, the decision was swift, and Ramanujan was granted a scholarship commencing on May 1, 1913. A stipulation in the scholarship required Ramanujan to submit quarterly reports detailing his research to the Board of Studies in Mathematics. Ramanujan wrote three such quarterly reports, dated August 5, 1913, November 7, 1913, and March 9, 1914, before he departed for England on March 17, 1914. Possibly the original reports remain at the University of Madras, but they evidently have either been lost or misplaced. Fortunately, in 1925, T. A. Satagopan made a handwritten copy of the reports on 51 foolscap pages. This copy was sent to G. H. Hardy along with a copy of Ramanujan's notebooks also made by Satagopan. G. N. Watson subsequently made a second handwritten copy of the reports. Both of these copies of the quarterly reports are now on file in the library of Trinity College, Cambridge.

The quarterly reports have never been published. However, Hardy used material from the reports as the basis for Chapter 11 of his book [20] on Ramanujan's work. We shall describe in detail the contents of these reports. In contrast to his notebooks which contain very few proofs, or even sketches, the quarterly reports offer several fairly detailed proofs. However, many of these proofs, especially those for the principal theorem, are formal and not rigorous. Nonetheless, Ramanujan's proofs are enormously interesting because they provide insight into how Ramanujan reasoned, and for this reason we shall describe Ramanujan's arguments. We also shall indicate, frequently with references to the literature, how Ramanujan's findings can be put on firm foundations.

The first two reports and a portion of the third are concerned with certain integral formulas which, in a sense, are interpolation formulas and which are connected with the theory of integral transforms. In discussing one of these formulas, Hardy [14, p. 150], [20, p. 15] remarks, "There is one particularly interesting formula ... of which he was especially fond and made continual use." In Chapter 11 of [20], Hardy further observes, about the aforementioned formulas, that Ramanujan "had not 'really' proved any of the formulae which I have quoted. It was impossible that he should have done so because the 'natural' conditions involve ideas of which he knew nothing in 1914, and which he had hardly absorbed before his death." The natural conditions to which Hardy refers are from the theory of functions of a complex variable. Indeed, as we shall momentarily see, the proofs given in the quarterly reports are not rigorous. After arriving in England, Ramanujan evidently learned, probably from Hardy, that his proofs were not sound. For in a paper published in 1915, after offering some beautiful integral evaluations, Ramanujan [13], [15, p. 57] remarks that, "My own proofs of the above results make use of a general formula, the truth of which depends on conditions which I have not yet investigated completely. A direct proof depending on Cauchy's theorem will be found in Mr. Hardy's note which follows this paper." These results will be expounded upon in the sequel.

Some of the principal formulas and their applications in the quarterly reports appear in Chapters 3 and 4 of the second notebook. Pages 180, 182, and 184 of the first notebook contain related "scratch" work. All of this relevant material in both notebooks can be found in Ramanujan's quarterly reports and/or his paper [13], [15, pp. 53–58]. We emphasize, however, that the quarterly reports contain many additional theorems and applications that have not been studied in the notebooks or Hardy's book [20].

One result in the reports which has not been discussed by Hardy and which does not appear in Ramanujan's published papers or his notebooks is a beautiful generalization of Frullani's integral theorem. This new theorem provides a powerful tool for evaluating many integrals and deserves more attention.

In addition to material on integral interpolation formulas and transforms, the third quarterly report contains results on orders of infinity, fractional composition of functions, and fractional differentiation. Each of these topics is treated only briefly. It might be recalled that in his first letter to Hardy, dated January 16, 1913, Ramanujan [15, p. xxiii] mentions that he has been reading Hardy's tract, *Orders of Infinity* [6].

We now discuss in turn each of the three quarterly reports. Each report is divided into several sections, and we shall adhere precisely to Ramanujan's divisions. The reports contain some minor misprints which we correct without comment.

1. The First Quarterly Report

In contrast to the second and third reports, the first report commences with a letter of introduction which we completely reproduce below.

From S. Ramanujan, Scholarship holder in mathematics.
To the Board of Studies in Mathematics.
Through The Registrar, University of Madras.
Gentlemen,

With reference to para. 2 of the University Registrar's letter no. 1631 dated the 9th April 1913, I beg to submit herewith my quarterly Progress Report for the quarter ended the 31st July, 1913.

The Progress Report is merely the exposition of a new theorem I have discovered in Integral Calculus. At present there are many definite integrals the values of which we know to be finite but still not possible of evaluation by the present known methods. This theorem will be an instrument by which at least some of the definite integrals whose values are at present not known can be evaluated. For instance, the integral treated in Ex(v) *note* Art. 5 in the paper, Mr. G. H. Hardy, M.A., F.R.S. of Trinity College, Cambridge, considers to be "new and interesting." Similarly the integral connected with the Besselian function of the *n*th order which at present requires many complicated manipulations to evaluate can be readily inferred from the theorem given in the paper. I have also utilised this theorem in definite integrals for the expansion of functions which can now be ordinarily done by Lagrange's, Burmann's, or Abel's theorems. For instance, the expansions marked as examples nos (3) and (4), Art. 6, in the second part of the paper.

The investigations I have made on the basis of this theorem are not all contained in the attached paper. There is ample scope for new and interesting results out of this theorem. This paper may be considered the first installment of the results I have got out of the theorem. Other new results based on the theorem I shall communicate in my later reports.

I beg to submit this, my maiden attempt, and I humbly request that the

Members of the Board will make allowance for any defect which they may notice to my want of usual training which is now undergone by college students and view sympathetically my humble effort in the attached paper.

I beg to remain,

Gentlemen

Your obedient servant
S. Ramanujan

1.1. The thrust of the quarterly reports is described at once. Suppose that $F(x)$ can be expanded in a Maclaurin series. Then Ramanujan asserts that the value of

$$I \equiv \int_0^\infty x^{n-1} F(x)\, dx$$

can be found from the coefficient of x^n in the expansion of $F(x)$. Conversely, he claims that if I can be determined, then the Maclaurin coefficients of $F(x)$ can be found.

1.2. Theorem I (Ramanujan's Master Theorem). *Suppose that, in some neighborhood of* $x = 0$,

$$F(x) = \sum_{k=0}^{\infty} \frac{\varphi(k)(-x)^k}{k!}.$$

Then

$$I = \Gamma(n)\varphi(-n). \tag{1.1}$$

Ramanujan's theorem needs some explanation. Generally, n is not an integer. Thus, Ramanujan is tactitly assuming that there exists a "natural," continuous extension of φ, defined initially on the set of nonnegative integers.

We shall first relate Ramanujan's proof and his adjoining discussion, and then we shall state Hardy's rigorous reformulation of Ramanujan's theorem. Observe that the proof given below is a slight generalization of that given in Section 11 of Chapter 4.

Ramanujan's proof. Recall Euler's integral representation of the gamma function

$$\int_0^\infty e^{-mx} x^{n-1}\, dx = m^{-n}\Gamma(n),$$

where $m, n > 0$. Let $m = r^k$ with $r > 0$, multiply both sides by $f^{(k)}(a)h^k/k!$, where f shall be specified later, and sum on k, $0 \le k < \infty$, to obtain

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)h^k}{k!} \int_0^\infty e^{-r^k x} x^{n-1}\, dx = \Gamma(n) \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(hr^{-n})^k}{k!}.$$

Next, expand $\exp(-r^k x)$, $0 \le k < \infty$, in its Maclaurin series, invert the order of summation and integration, invert the order of summation, and apply Taylor's theorem to deduce that

$$\int_0^\infty x^{n-1} \sum_{j=0}^\infty \frac{f(hr^j + a)(-x)^j}{j!} dx = \Gamma(n) f(hr^{-n} + a). \tag{1.2}$$

Now define $f(hr^m + a) = \varphi(m)$, where m is real, and a, h, and r are regarded as constants. Then (1.2) may be rewritten in the form

$$\int_0^\infty x^{n-1} \sum_{j=0}^\infty \frac{\varphi(j)(-x)^j}{j!} dx = \Gamma(n)\varphi(-n),$$

which completes Ramanujan's proof.

Ramanujan was evidently quite fond of this very clever, original technique, and he employed it in many contexts. See Chapter 4 for several additional illustrations.

1.3. Of course, Ramanujan's procedure is fraught with numerous difficulties. Ramanujan asserts that his proof is legitimate with just four simple assumptions: (1) $F(x)$ can be expanded in a Maclaurin series; (2) $F(x)$ is continuous on $(0, \infty)$; (3) $n > 0$; and (4) $x^n F(x)$ tends to 0 as x tends to ∞. He remarks that the fourth condition can be relaxed if, for example, $F(x)$ is a circular or Bessel function. However, these four conditions are not nearly strong enough.

In preparation for stating Hardy's version of Theorem I, we need to introduce some notation. Let $s = \sigma + it$ with σ and t both real. Let $H(\delta) = \{s: \sigma \geq -\delta\}$, where $0 < \delta < 1$. Suppose that $\psi(s)$ is analytic on $H(\delta)$ and that there exist constants C, P, and A with $A < \pi$ such that

$$|\psi(s)| \leq Ce^{P\sigma + A|t|}, \tag{1.3}$$

for all $s \in H(\delta)$. For $x > 0$ and $0 < c < \delta$, define

$$\Psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi s)} \psi(-s) x^{-s} ds.$$

If $0 < x < e^{-P}$, an application of the residue theorem yields (see Hardy's book [20, p. 189])

$$\Psi(x) = \sum_{k=0}^\infty \psi(k)(-x)^k.$$

Theorem (Hardy [20, pp. 189, 190]). *Let ψ and Ψ satisfy the conditions set forth in the preceding paragraph. Suppose that $0 < \sigma < \delta$. Then*

$$\int_0^\infty \Psi(x) x^{s-1} dx = \frac{\pi}{\sin(\pi s)} \psi(-s). \tag{1.4}$$

Formula (1.4) yields (1.1) upon replacing $\psi(s)$ by $\varphi(s)/\Gamma(s+1)$. In Hardy's book [20], formulas (1.1) and (1.4) are (B) and (A), respectively, on p. 186.

Ramanujan devotes the remainder of the first quarterly report, all of the second report, and a large portion of the third to giving examples and applications of his Master Theorem. For the individual examples we shall determine if the hypotheses of Hardy's theorem are satisfied. However, for brevity, we shall generally not recast Ramanujan's theorems that are derived from his Master Theorem into rigorous formulations. Thus, let it be tacitly assumed in the sequel that Ramanujan's theorems can be placed on solid ground if the relevant functions satisfy the hypotheses of Hardy's theorem.

1.4. Ramanujan next briefly indicates some of the kinds of functions to which his Master Theorem is applicable.

1.5. Examples. (i) This first example is mentioned by Hardy in his book [20, p. 188]. Let $m, n > 0$ with $m < n$. Letting $x = y^{1/n}$, we find that

$$\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{1}{n}\int_0^\infty \frac{y^{m/n-1}}{1+y}\,dy.$$

Expanding $1/(1+y)$ into a geometric series, we see that, in the notation of the Master Theorem, $\varphi(s) = \Gamma(s+1)$. Hardy's hypotheses are easily seen to be satisfied, and so (1.1) gives

$$\begin{aligned}\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx &= \frac{1}{n}\Gamma\left(\frac{m}{n}\right)\varphi\left(-\frac{m}{n}\right)\\ &= \frac{1}{n}\Gamma\left(\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right) = \frac{\pi}{n\sin(\pi m/n)},\end{aligned}$$

which is a familiar result.

(ii) The second example is Corollary 5, Section 11 of Chapter 4. Let $m, n > 0$ and set $x = y/(1+y)$ to obtain

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty y^{m-1}(1+y)^{-m-n}\,dy.$$

From the binomial series,

$$(1+y)^{-r} = \sum_{k=0}^\infty \frac{\Gamma(k+r)}{\Gamma(r)k!}(-y)^k, \qquad |y| < 1, \tag{1.5}$$

we find that $\varphi(s) = \Gamma(s+m+n)/\Gamma(m+n)$. By Stirling's formula (I6), the hypotheses of Hardy's theorem are readily verified. Hence, Ramanujan's Master Theorem yields the following well-known representation of the beta-function $B(m, n)$,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \Gamma(m)\varphi(-m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{1.6}$$

(iii) Let $p>0$ and $0<n<1$. Letting $x=\sqrt{y}$, we find that

$$\int_0^\infty x^{n-1}\cos(px)\,dx=\tfrac{1}{2}\int_0^\infty y^{n/2-1}\cos(p\sqrt{y})\,dy.$$

Expanding $\cos(p\sqrt{y})$ into a Maclaurin series, we find that, in Hardy's notation, $\psi(s)=p^{2s}/\Gamma(2s+1)$. By Stirling's formula, we deduce that, in the notation (1.3), $A=\pi+\varepsilon$, for any $\varepsilon>0$. Hence, with no justification, we proceed, as did Ramanujan, to conclude that

$$\int_0^\infty x^{n-1}\cos(px)\,dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\varphi(-\tfrac{1}{2}n)$$

$$=\frac{\Gamma(\frac{1}{2}n)\Gamma(1-\frac{1}{2}n)}{2p^n\Gamma(1-n)}=\frac{\Gamma(n)\cos(\frac{1}{2}\pi n)}{p^n}. \tag{1.7}$$

Now, in fact, Ramanujan's evaluation is, indeed, correct (Gradshteyn and Ryzhik [1, p. 421]).

Ramanujan next shows that

$$\int_0^\infty x^{n-1}\sin(px)\,dx=\frac{\Gamma(n)\sin(\frac{1}{2}n)}{p^n},\qquad |n|<1, \tag{1.8}$$

by replacing n by $n-1$ in (1.7) and differentiating both sides with respect to p. This procedure is invalid because the integral obtained from (1.7) by replacing n by $n-1$ does not converge in the domain $|n|<1$ of (1.8). Nonetheless, (1.8) is still correct (Gradshteyn and Ryzhik [1, p. 420]). If we follow the procedure employed by Ramanujan in obtaining (1.7), we also deduce (1.8). As before, this method is not justified.

(iv) Recall that the ordinary Bessel function $J_n(x)$ of order n is defined for all real values of n and x by

$$J_n(x)=\sum_{k=0}^\infty\frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{2k+n}.$$

Consider

$$\int_0^\infty x^{p-n-1}J_n(x)\,dx=\int_0^\infty y^{p/2-1}\sum_{k=0}^\infty\frac{(-y)^k}{k!\,\Gamma(n+k+1)2^{n+2k+1}}\,dy.$$

Clearly, we need to require that $p>0$. Also, from the asymptotic expansion of $J_n(x)$ as x tends to ∞ (see Whittaker and Watson's text [1, p. 368]), the integrals above converge if $p<n+\frac{3}{2}$. In Hardy's notation, $1/\psi(s)=2^{n+1+2s}\Gamma(s+1)\Gamma(n+s+1)$ and $A=\pi+\varepsilon$, for any $\varepsilon>0$, by Stirling's formula. Thus, Hardy's theorem is inapplicable. However, formally applying Ramanujan's Master Theorem, we find that

$$\int_0^\infty x^{p-n-1}J_n(x)\,dx=\Gamma(\tfrac{1}{2}p)\varphi(-\tfrac{1}{2}p)=\frac{2^{p-n-1}\Gamma(\frac{1}{2}p)}{\Gamma(n+1-\frac{1}{2}p)},$$

where $0 < p < n + \frac{3}{2}$. Despite the faulty procedure, this result is again correct (Gradshteyn and Ryzhik [1, p. 684]; Watson [3, p. 391]).

(v) For $0 < q < 1$, define

$$(a;\, q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$$

and

$$I = \int_0^\infty \frac{t^{n-1}\, dt}{(-t;\, q)_\infty}, \qquad n > 0.$$

This integral is one to which Ramanujan refers in his prefatory comments. It is actually the particular case $a = 0$ of the more general integral

$$f(a) = \int_0^\infty \frac{t^{n-1}(-at;\, q)_\infty\, dt}{(-t;\, q)_\infty}, \qquad |a| < q^n,$$

discussed by him in [13], [15, pp. 53–58]. As we inferred in our introduction, Ramanujan used the Master Theorem to evaluate $f(a)$. In a paper [8], [19, pp. 594–597] immediately following Ramanujan's paper [13], Hardy evaluates $f(a)$ by contour integration. R. Askey [1] has made a penetrating study of $f(a)$ and has shown that if $n = x$ and $a = q^{x+y}$, then $f(a)$ is the natural q-analogue of the beta function $B(x, y)$ discussed in Example (ii).

We now describe Ramanujan's proof which is also briefly sketched by Hardy [20, p. 194]. Writing

$$\frac{1}{(-t;\, q)_\infty} = \sum_{k=0}^{\infty} \psi(k)(-t)^k, \qquad |t| < 1,$$

and using the fact that $(1 + t)(-qt;\, q)_\infty = (-t;\, q)_\infty$, we easily derive the recursion formula

$$\psi(k) = \frac{\psi(k-1)}{1 - q^k}, \qquad k \geq 1.$$

Since $\psi(0) = 1$, an inductive argument shows that

$$\psi(k) = \frac{(q^{k+1};\, q)_\infty}{(q;\, q)_\infty}, \qquad k \geq 0,$$

which satisfies the hypotheses of Hardy's theorem. Hence, by Ramanujan's Master Theorem, or Hardy's theorem,

$$I = \frac{\pi}{\sin(\pi n)}\, \psi(-n) = \frac{\pi(q^{1-n};\, q)_\infty}{\sin(\pi n)(q;\, q)_\infty}. \tag{1.9}$$

Example (a). If $n = 1$, (1.9) becomes

$$\int_0^\infty \frac{dt}{(-t;\, q)_\infty} = \lim_{n \to 1} \frac{\pi(1 - q^{1-n})}{\sin(\pi n)} = -\operatorname{Log} q.$$

Example (b). Let $n=2$ and replace t by $\sqrt{t}$ in (1.9) to find that

$$\int_0^\infty \frac{dt}{(-\sqrt{t};q)_\infty} = \lim_{n\to 2} \frac{2\pi(1-q^{1-n})(1-q^{2-n})}{\sin(\pi n)}$$

$$= \frac{2(q-1)\operatorname{Log} q}{q}.$$

Example (c). Letting $n=3$ and replacing t by $t^{1/3}$ in (1.9), we deduce that

$$\int_0^\infty \frac{dt}{(-t^{1/3};q)_\infty} = \lim_{n\to 3} \frac{3\pi(1-q^{1-n})(1-q^{2-n})(1-q^{3-n})}{\sin(\pi n)}$$

$$= -\frac{3(1-q)(1-q^2)\operatorname{Log} q}{q^3}.$$

Example (d). Let $n=\frac{1}{2}$, $q=a^2$, and $t=x^2$ in (1.9) to discover the elegant identity

$$\int_0^\infty \frac{dx}{(-x^2;a^2)_\infty} = \frac{\pi}{2}\prod_{k=1}^\infty \frac{1-a^{2k-1}}{1-a^{2k}},$$

which was first posed as a problem by Ramanujan [6], [15, p. 326]. Ramanujan also gives this application in [13], [15, p. 58] and further remarks that

$$\sum_{k=1}^\infty a^{k(k-1)/2} = \prod_{k=1}^\infty \frac{1-a^{2k}}{1-a^{2k-1}}.$$

This result is due to Gauss, and a proof can be found in G. E. Andrews' book [1, p. 23].

(vi) In the last example of this section, Ramanujan shows that if $a>0$, $m<1$, and $m+n>0$, then

$$\int_0^\infty \frac{\Gamma(x+a)\,dx}{\Gamma(x+a+n+1)x^m} = \frac{\pi\csc(\pi m)}{\Gamma(n+1)}\sum_{k=0}^\infty \binom{n}{k}\frac{(-1)^k}{(a+k)^m}. \tag{1.10}$$

The conditions on a, m, and n are needed for the convergence of the integral. Also, since

$$\binom{n}{k} \sim \frac{(-1)^k}{\Gamma(-n)}k^{-n-1},$$

as k tends to ∞, if n is not a nonnegative integer, the condition $m+n>0$ insures the convergence of the series on the right side. We have not been able to find (1.10) in the literature.

We now present Ramanujan's derivation. From (1.6) and (1.5), for $x+a$, $n+1>0$,

$$\begin{aligned}\frac{\Gamma(x+a)\Gamma(n+1)}{\Gamma(x+a+n+1)} &= \int_0^1 t^{x+a-1}(1-t)^n\,dt \\ &= \int_0^1 t^{x+a-1}\sum_{k=0}^{\infty}\binom{n}{k}(-t)^k\,dt \\ &= \sum_{k=0}^{\infty}\binom{n}{k}\frac{(-1)^k}{x+a+k} \\ &= \sum_{k=0}^{\infty}\binom{n}{k}\frac{(-1)^k}{a+k}\sum_{j=0}^{\infty}\left(\frac{-x}{a+k}\right)^j \\ &= \sum_{j=0}^{\infty}\psi(j)(-x)^j, \end{aligned} \tag{1.11}$$

provided that $|x|<a$, where

$$\psi(s)=\sum_{k=0}^{\infty}\binom{n}{k}\frac{(-1)^k}{(a+k)^{s+1}}, \qquad s+n+1>0.$$

Now multiply the left side of (1.11) by x^{-m} and integrate over $0\le x<\infty$. It is easily checked that the hypotheses of Hardy's theorem are satisfied with $\psi(s)$ as given above. Hence, by (1.4),

$$\begin{aligned}\int_0^\infty \frac{\Gamma(x+a)\Gamma(n+1)\,dx}{\Gamma(x+a+n+1)x^m} &= \frac{\pi\psi(m-1)}{\sin\{\pi(1-m)\}} \\ &= \pi\csc(\pi m)\sum_{k=0}^{\infty}\binom{n}{k}\frac{(-1)^k}{(a+k)^m},\end{aligned}$$

from which (1.10) is apparent.

1.6. In the final section of the first report, Ramanujan derives certain expansions for four functions by assuming that a type of converse theorem to the Master Theorem holds. More specifically, he determines a power series for the integrand from the value of the integral. In fact, Ramanujan's converse to the Master Theorem follows from the inversion formula for Mellin transforms. Although Ramanujan proceeded formally, all of the results that he obtains are, indeed, correct. We shall not only give Ramanujan's argument but also indicate a correct proof in each case.

(i) We first want to expand $(2/(1+\sqrt{1+4x}))^n$ in powers of x when $n>0$. Let $0<p<n/2$ and consider

$$I\equiv\int_0^\infty x^{p-1}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx.$$

Setting $x = y + y^2$ and then $y = z/(1-z)$, we find that

$$I = \int_0^\infty y^{p-1}(1+y)^{p-n-1}(1+2y)\,dy$$

$$= \int_0^1 z^{p-1}(1-z)^{n-2p-1}(1+z)\,dz$$

$$= \frac{\Gamma(p)\Gamma(n-2p)}{\Gamma(n-p)} + \frac{\Gamma(p+1)\Gamma(n-2p)}{\Gamma(n-p+1)} = \frac{n\Gamma(p)\Gamma(n-2p)}{\Gamma(n-p+1)},$$

where we have employed (1.6). Hence, in the notation of (1.1), $\varphi(p) = n\Gamma(n+2p)/\Gamma(n+p+1)$. Ramanujan thus concludes that

$$\left(\frac{2}{1+\sqrt{1+4x}}\right)^n = n\sum_{k=0}^\infty \frac{\Gamma(n+2k)(-x)^k}{\Gamma(n+k+1)k!}, \qquad |x| \le \tfrac{1}{4}. \tag{1.12}$$

This result is, indeed, correct. In fact, the restriction that n be positive is unnecessary. Of course, one could also establish (1.12) by applying Taylor's theorem. Equality (1.12) can also be found in Corollary 1, Section 14 of Chapter 3, where a short proof, based on Example (iv) below, is given.

(ii) We next wish to expand $(x+\sqrt{1+x^2})^{-n}$ in ascending powers of x when $n > 0$. Letting $x + \sqrt{1+x^2} = 1/\sqrt{y}$, Ramanujan considers, for $0 < p < n$,

$$\int_0^\infty \frac{x^{p-1}\,dx}{(x+\sqrt{1+x^2})^n} = \frac{1}{2^{p+1}}\int_0^1 (1-y)^{p-1}y^{(n-p)/2}(1+1/y)\,dy$$

$$= \frac{n\Gamma(p)\Gamma(\frac{1}{2}\{n-p\})}{2^{p+1}\Gamma(\frac{1}{2}\{n+p\}+1)},$$

by (1.6). In the notation of the Master Theorem,

$$\varphi(p) = \frac{n2^{p-1}\Gamma(\frac{1}{2}\{n+p\})}{\Gamma(\frac{1}{2}\{n-p\}+1)}.$$

Hence, Ramanujan concludes that

$$(x+\sqrt{1+x^2})^{-n} = n\sum_{k=0}^\infty \frac{2^{k-1}\Gamma(\frac{1}{2}\{n+k\})(-x)^k}{\Gamma(\frac{1}{2}\{n-k\}+1)k!}, \qquad |x| \le 1. \tag{1.13}$$

The expansion (1.13) is, in fact, valid for all complex values of n. Moreover, (1.13) can be found in Corollary 2, Section 14 of Chapter 3, where a short proof based upon Example (iv) below can be found.

Now replace n by in and x by ix in (1.13). Recalling that

$$\sin^{-1} x = -i\,\mathrm{Log}\,\{ix + \sqrt{1-x^2}\},$$

where principal branches are taken, we deduce from (1.13) that

$$e^{n\sin^{-1}x} = 1 + nx + \sum_{k=2}^{\infty} \frac{b_k(n)x^k}{k!}, \qquad |x| \le 1,$$

where, for $k \ge 2$,

$$b_k(n) = \begin{cases} n^2(n^2+2^2)(n^2+4^2)\cdots(n^2+(k-2)^2), & \text{if } k \text{ is even,} \\ n(n^2+1^2)(n^2+3^2)\cdots(n^2+(k-2)^2), & \text{if } k \text{ is odd.} \end{cases}$$

(iii) Let $a \ge 0$ and let x be the unique positive solution to the equation $\operatorname{Log} x = -ax$. For each positive number n, we want to expand x^n in ascending powers of a. Letting $0 < p < n$, $a = -(\operatorname{Log} x)/x$, and then $x = e^{-y}$, Ramanujan finds that

$$\begin{aligned} \int_0^\infty a^{p-1}x^n\,da &= \int_0^1 \left(-\frac{\operatorname{Log} x}{x}\right)^{p-1} x^n \frac{1-\operatorname{Log} x}{x^2}\,dx \\ &= \int_0^\infty y^{p-1}(1+y)e^{-y(n-p)}\,dy \\ &= \frac{n\Gamma(p)}{(n-p)^{p+1}}. \end{aligned}$$

Thus, in the notation of the Master Theorem, $\varphi(p) = n(n+p)^{p-1}$. Therefore, Ramanujan concludes that

$$x^n = n \sum_{k=0}^{\infty} \frac{(n+k)^{k-1}(-a)^k}{k!}. \tag{1.14}$$

Using Stirling's formula (I6), one can show that the infinite series in (1.14) converges for $0 \le a \le 1/e$.

In fact, (1.14) is valid for every real number n and $|a| \le 1/e$. The expansion (1.14) can be found in Chapter 3, Entry 13, where a rigorous proof has been given.

(iv) Consider the trinomial equation

$$aqx^p + x^q = 1, \tag{1.15}$$

where $a > 0$ and $0 < q < p$. We shall find an expansion for x^n in nonnegative powers of a, where n is any positive real number and x is a particular root of (1.15). Ramanujan's derivation is briefly presented in Hardy's book [20, pp. 194, 195].

Choose r so that $0 < pr < n$. Making the substitutions $a = (1-y)/(qy^{p/q})$ and $x = y^{1/q}$, we find that

$$\begin{aligned} \int_0^\infty a^{r-1}x^n\,da &= \frac{1}{q^r}\int_0^1 y^{n/q}\left(\frac{1-y}{y^{p/q}}\right)^{r-1}\left\{\frac{p(1-y)}{qy^{p/q+1}} + y^{-p/q}\right\}dy \\ &= \frac{p}{q^{r+1}}\int_0^1 y^{(n-pr)/q-1}(1-y)^r\,dy \end{aligned}$$

$$+\frac{1}{q^r}\int_0^1 y^{(n-pr)/q}(1-y)^{r-1}\,dy$$
$$=\frac{n\Gamma(r)\Gamma(\{n-pr\}/q)}{q^{r+1}\Gamma(\{n-pr\}/q+r+1)},$$

by (1.6). Thus, in the notation of the Master Theorem,

$$\varphi(r)=\frac{nq^{r-1}\Gamma(\{n+pr\}/q)}{\Gamma(\{n+pr\}/q-r+1)}.$$

Hence, Ramanujan concludes that

$$x^n=\frac{n}{q}\sum_{k=0}^{\infty}\frac{\Gamma(\{n+pk\}/q)(-qa)^k}{\Gamma(\{n+pk\}/q-k+1)k!}. \tag{1.16}$$

The expansion (1.16) is actually valid for all real numbers n, p, and q, and for complex a with

$$|a|\leq|p|^{-p/q}|p-q|^{(p-q)/q}. \tag{1.17}$$

Equality (1.16) is stated in Chapter 3, Entry 14, where a legitimate proof is given.

Formulas (1.14) and (1.16) have a very long history. The latter was first discovered in 1758 by Lambert [1], while the former was initially found by Euler [4], [6] in 1779. In 1770, Lagrange [1] discovered his famous "Lagrange inversion formula," which can be found in Whittaker and Watson's treatise [1, p. 133], and which is sometimes called the Lagrange–Bürmann theorem, and as an application derived (1.16). Recall that in the introduction to the first report, Ramanujan makes a reference to theorems of Lagrange and Bürmann. Ramanujan evidently learned about these theorems from Carr's *Synopsis* [1, pp. 278–282], where expansions similar to (1.12) and (1.13) are given as applications. Pólya and Szegö [1, p. 146] have offered (1.14) and (1.16) as exercises illustrating the Lagrange inversion formula. The derivations of (1.14) and (1.16) in Chapter 3 are developed *ab initio*, however.

2. The Second Quarterly Report

2.1. Ramanujan commences the second quarterly report with two further applications of his Master Theorem.

Example (a). Let $0<r<1$ and suppose that m is real. Then

$$\int_0^\infty x^{r-1}\sum_{k=0}^{\infty}\frac{(k+1)^m(-x)^k}{k!}\,dx=\Gamma(r)(1-r)^m. \tag{2.1}$$

In the notation of Ramanujan's Master Theorem, $\varphi(s)=(s+1)^m$, and so Hardy's hypotheses are readily seen to be satisfied. Hence, (2.1) follows immediately from the Master Theorem. This example is also mentioned in Hardy's book [20, p. 193].

Although (2.1) is not found in his notebooks, Ramanujan thoroughly discusses the integrand of (2.1) when m is a nonnegative integer in Chapter 3. In fact,

$$\sum_{k=0}^{\infty} \frac{(k+1)^m(-x)^k}{k!} = -\frac{e^{-x}\varphi_{m+1}(-x)}{x},$$

where $\varphi_n(x)$ is a polynomial of degree n called the nth single variable Bell polynomial.

Ramanujan illustrates (2.1) with four examples which we need not record in detail here. The examples are for $m = 1$ and $r > 0$, $m = 2$ and $r > 0$, $m = \pi$ and $r = \frac{1}{9}$, and m real and $r = \frac{1}{2}$. In the former two cases, the integrands may be expressed in terms of elementary functions.

Example (b). Ramanujan next derives several related expansions by appealing to a converse of his Master Theorem, as in the first report. We first expand e^{ax} in ascending powers of $y = e^{-bx}\sinh(cx)/c$, where $a, b \geq 0$ and $c > 0$.

Choose $n > 0$ so that $a + n(b - c) > 0$. Letting $y = e^{-bx}\sinh(cx)/c$, consider

$$\begin{aligned} I &= \int_0^\infty y^{n-1} e^{-ax}\,dy \\ &= \int_0^\infty \left(\frac{e^{-bx}\sinh(cx)}{c}\right)^{n-1} e^{-ax}\frac{(c-b)e^{(c-b)x} + (c+b)e^{-(c+b)x}}{2c}\,dx \\ &= \frac{c-b}{(2c)^n}\int_0^\infty e^{-\{a+n(b-c)\}x}(1 - e^{-2cx})^{n-1}\,dx \\ &\quad + \frac{c+b}{(2c)^n}\int_0^\infty e^{-\{a+n(b-c)+2c\}x}(1 - e^{-2cx})^{n-1}\,dx. \end{aligned}$$

Since for $p, q > 0$,

$$\begin{aligned} \int_0^\infty e^{-px}(1 - e^{-qx})^{n-1}\,dx &= \frac{1}{q}\int_0^1 t^{p/q-1}(1-t)^{n-1}\,dt \\ &= \frac{\Gamma(n)\Gamma(p/q)}{q\Gamma(n + p/q)}, \end{aligned}$$

by (1.6), we find that

$$\begin{aligned} I &= \frac{(c-b)\Gamma(n)\Gamma\left(\dfrac{a+n(b-c)}{2c}\right)}{(2c)^{n+1}\Gamma\left(n + \dfrac{a+n(b-c)}{2c}\right)} + \frac{(c+b)\Gamma(n)\Gamma\left(\dfrac{a+n(b-c)+2c}{2c}\right)}{(2c)^{n+1}\Gamma\left(n + \dfrac{a+n(b-c)+2c}{2c}\right)} \\ &= \frac{a\Gamma(n)\Gamma\left(\dfrac{a+n(b-c)}{2c}\right)}{(2c)^{n+1}\Gamma\left(\dfrac{a+n(b+c)}{2c} + 1\right)}. \end{aligned}$$

Hence, Ramanujan concludes from the Master Theorem that

$$\varphi(s)=\frac{a(2c)^{s-1}\Gamma\left(\dfrac{a+s(c-b)}{2c}\right)}{\Gamma\left(\dfrac{a-s(b+c)}{2c}+1\right)}$$

and

$$e^{-ax}=a\sum_{k=0}^{\infty}\frac{\Gamma\left(\dfrac{a+k(c-b)}{2c}\right)(2c)^{k-1}}{\Gamma\left(\dfrac{a-k(b+c)}{2c}+1\right)k!}\left(-\frac{e^{-bx}\sinh(cx)}{c}\right)^{k}. \qquad (2.2)$$

Now, in fact, (2.2) is really a special case of Example (iv) at the end of the first report. In the notation of that example, replace x by e^{-x}, let $a=e^{-bx}\sinh(cx)/c$, let $p=c-b$, and let $q=2c$. Then a brief calculation shows that (1.15) is satisfied. Letting n (in Example (iv)) $=a$, we find that (1.16) reduces to (2.2). Furthermore, by (1.17), the representation in (2.2) converges for

$$\left|\frac{e^{-bx}\sinh(cx)}{c}\right|\leq|b-c|^{(b-c)/(2c)}|b+c|^{-(b+c)/(2c)}. \qquad (2.3)$$

Ramanujan next derives an expansion for e^{ax} in ascending powers of $e^{-bx}\sin(cx)/c$. First observe that $e^{-bx}\sin(cx)$ is an increasing function on $[0, x_0]$, where $x_0=(1/c)\tan^{-1}(c/b)$, the point at which $e^{-bx}\sin(cx)$ achieves its maximum for $x\geq 0$. Thus, at the outset, we restrict our attention to $0\leq x\leq x_0$. Ramanujan proceeds formally and simply by replacing a by $-a$ and c by ci in (2.2). Although in Chapter 3 it is assumed that p and q are real, the proof of (1.16) can be easily generalized to allow for complex values of p and q. Thus, proceeding as in our derivation of (2.2), we find with Ramanujan that

$$\begin{aligned} e^{ax}&=a\sum_{k=0}^{\infty}\frac{\Gamma\left(\dfrac{-a+k(ci-b)}{2ci}\right)(-2ci)^{k-1}}{\Gamma\left(\dfrac{-a-k(ci+b)}{2ci}+1\right)k!}\left(\frac{e^{-bx}\sin(cx)}{c}\right)^{k}\\ &=1+\frac{ae^{-bx}\sin(cx)}{c}+\sum_{k=2}^{\infty}\frac{d^k}{k!}\left(\frac{e^{-bx}\sin(cx)}{c}\right)^{k}, \end{aligned} \qquad (2.4)$$

provided that the series converges, where, for $k\geq 2$,

$$d_k=\begin{cases} a(a+kb)\{(a+kb)^2+(2c)^2\}\{(a+kb)^2+(4c)^2\}\cdots\{(a+kb)^2+(k-2)^2c^2\}, & \text{if } k \text{ is even},\\ a\{(a+kb)^2+c^2\}\{(a+kb)^2+(3c)^2\}\cdots\{(a+kb)^2+(k-2)^2c^2\}, & \text{if } k \text{ is odd}. \end{cases} \qquad (2.5)$$

By Stirling's formula (I6) (compare also with (2.3)), the series on the right side of (2.4) converges when

$$\left|\frac{e^{-bx}\sin(cx)}{c}\right| \le |(b-ci)^{(b-ci)/(2ci)}(b+ci)^{-(b+ci)/(2ci)}|$$

$$= (b^2+c^2)^{-\frac{1}{2}} \exp\left\{-\frac{b}{c}\tan^{-1}\left(\frac{c}{b}\right)\right\}. \tag{2.6}$$

A short calculation shows that when $x = x_0$ there is equality above. Thus, since $e^{-bx}\sin(cx)$ is increasing on $[0, x_0]$, (2.4) is valid for $0 \le x \le x_0$.

2.2. In this section, Ramanujan derives four corollaries of (2.4).

Letting $b = 0$ in (2.4)–(2.6), we find that, for $0 \le x \le \pi/(2c)$,

$$e^{ax} = 1 + \frac{a\sin(cx)}{c} + \sum_{k=2}^{\infty} \frac{u_k}{k!}\left(\frac{\sin(cx)}{c}\right)^k,$$

where, for $k \ge 2$,

$$u_k = \begin{cases} a^2\{a^2+(2c)^2\}\{a^2+(4c)^2\}\cdots\{a^2+(k-2)^2c^2\}, & \text{if } k \text{ is even,} \\ a\{a^2+c^2\}\{a^2+(3c)^2\}\cdots\{a^2+(k-2)^2c^2\}, & \text{if } k \text{ is odd.} \end{cases}$$

Letting c tend to 0 in (2.4)–(2.6), we find that, for $0 \le x \le 1/b$,

$$e^{ax} = a\sum_{k=0}^{\infty} \frac{(a+kb)^{k-1}x^k e^{-kbx}}{k!}. \tag{2.7}$$

Ramanujan evidently first learned of this well-known formula in Carr's book [1, p. 282]. The expansion (2.7) can be used to establish a more general theorem due to Abel, found in Carr's *Synopsis* [1, p. 282], to which Ramanujan refers in his introductory letter to these reports. In a slightly different form, (2.7) may be found as (13.8) in Chapter 3. Formula (2.7) is stated as a problem in Bromwich's text [1, p. 160] and is mentioned by Hardy [20, p. 194].

For the third illustration, Ramanujan replaces a by ai and b by bi in (2.4) and equates real parts in the resulting equality to obtain an expansion of $\cos(ax)$ in ascending powers of $\sin(cx)/c$. This procedure is not really justified because the primary expansion (1.16) was derived under the assumption that n be real. However, we can deduce Ramanujan's expansion directly from (1.16) in the following manner. Let $a = ie^{-ibx}\sin(cx)/c$, $p = c - b$, and $q = 2c$ in (1.15), where $b \ge 0$ and $c > 0$. With x replaced by e^{-ix}, we see that, indeed, (1.15) is satisfied. Since we shall expand in powers of $e^{-ibx}\sin(cx)$, we need to restrict x, and so we add the assumption,

$$|x| \le \frac{\pi}{2(b+c)}. \tag{2.8}$$

Thus, from (1.16), with n replaced by $-a$, and (1.17), we find that

$$e^{iax} = a \sum_{k=0}^{\infty} \frac{\Gamma\left(\dfrac{-a+(c-b)k}{2c}\right)(-2c)^{k-1}}{\Gamma\left(\dfrac{-a-(c+b)k}{2c}+1\right)k!} \left(\frac{ie^{-ibx}\sin(cx)}{c}\right)^k, \tag{2.9}$$

provided that

$$\left|\frac{\sin(cx)}{c}\right| \leq |b-c|^{(b-c)/(2c)}|b+c|^{-(b+c)/(2c)}. \tag{2.10}$$

Equating real parts in (2.9), we deduce Ramanujan's expansion

$$\cos(ax) = 1 + \frac{a\sin(bx)\sin(cx)}{c} + \sum_{k=2}^{\infty} \frac{v_k}{k!}\left(\frac{\sin(cx)}{c}\right)^k,$$

where, for $k \geq 2$,

$$v_k = \begin{cases} (-1)^{k/2}a(a+kb)\{(a+kb)^2-(2c)^2\}\{(a+kb)^2-(4c)^2\}\cdots\{(a+kb)^2 \\ \qquad -(k-2)^2c^2\}\cos(kbx), \text{ if } k \text{ is even}, \\ (-1)^{(k-1)/2}a\{(a+kb)^2-c^2\}\{(a+kb)^2-(3c)^2\}\cdots\{(a+kb)^2 \\ \qquad -(k-2)^2c^2\}\sin(kbx), \text{ if } k \text{ is odd}, \end{cases}$$

provided that x satisfies (2.8) and (2.10). Of course, a similar expansion for $\sin(ax)$ may be derived.

Lastly, return to (2.4) and equate the coefficients of a on both sides to deduce that

$$x = \frac{e^{-bx}\sin(cx)}{c} + \sum_{k=2}^{\infty} \frac{w_k}{k!}\left(\frac{e^{-bx}\sin(cx)}{c}\right)^k, \tag{2.11}$$

where, for $k \geq 2$,

$$w_k = \begin{cases} kb\{(kb)^2+(2c)^2\}\{(kb)^2+(4c)^2\}\cdots\{(kb)^2 \\ \qquad +(k-2)^2c^2\}, \quad \text{if } k \text{ is even}, \\ \{(kb)^2+c^2\}\{(kb)^2+(3c)^2\}\cdots\{(kb)^2 \\ \qquad +(k-2)^2c^2\}, \quad \text{if } k \text{ is odd}, \end{cases} \tag{2.12}$$

provided that $0 \leq x \leq x_0$. Similar, but more complicated, expansions can be deduced for higher powers of x.

2.3. Ramanujan next gives a beautiful application of (2.11) to approximating the roots of the transcendental equation

$$\sum_{k=0}^{\infty} \frac{(-x^3)^k}{(3k)!} = \frac{e^{-x}+e^{-\omega x}+e^{-\omega^2 x}}{3}$$

$$= \frac{e^{-x}+2e^{x/2}\cos(x\sqrt{3}/2)}{3} = 0,$$

where ω is a primitive cube root of unity. Thus,

$$\cos\left(\frac{x\sqrt{3}}{2}\right) = -\tfrac{1}{2}e^{-3x/2}. \tag{2.13}$$

By examining the graphs of $y=\cos(x\sqrt{3}/2)$ and $y=-\frac{1}{2}\exp(-3x/2)$, we see that there are an infinite number of positive roots of (2.13). Furthermore, these roots are very near the roots $\pi(2n+1)/\sqrt{3}$ of $\cos(x\sqrt{3}/2)$, where n is a nonnegative integer. Setting $x=\pi(2n+1)/\sqrt{3}-z$, we transform (2.13) into

$$(-1)^n \sin\left(\frac{z\sqrt{3}}{2}\right) = -\tfrac{1}{2}e^{-\pi(2n+1)\sqrt{3}/2+3z/2},$$

and then with $h=\exp(-\pi(2n+1)\sqrt{3}/2)$, the equality above may be written as

$$\tfrac{1}{2}(-1)^{n+1}h = e^{-3z/2}\sin\left(\frac{z\sqrt{3}}{2}\right).$$

We now apply (2.11) with $b=\frac{3}{2}$ and $c=\sqrt{3}/2$ to expand z in powers of h. Hence,

$$z = \frac{(-1)^{n+1}h}{\sqrt{3}} + \sum_{k=2}^{\infty}\frac{w_k}{k!}\left(\frac{(-1)^{n+1}h}{\sqrt{3}}\right)^k,$$

or, by (2.12),

$$\begin{aligned} x = \frac{\pi(2n+1)}{\sqrt{3}} - z = \frac{\pi(2n+1)}{\sqrt{3}} &+ \frac{(-1)^n}{\sqrt{3}}\Bigg(h + \frac{7}{3!}h^3 + \frac{19\cdot 21}{5!}h^5 \\ &+ \frac{37\cdot 39\cdot 43}{7!}h^7 + \frac{61\cdot 63\cdot 67\cdot 73}{9!}h^9 \\ &+ \frac{91\cdot 93\cdot 97\cdot 103\cdot 111}{11!}h^{11} + \cdots\Bigg) \\ &- \tfrac{1}{2}\Bigg(h^2 + \frac{13}{3!}h^4 + \frac{28\cdot 31}{5!}h^6 + \frac{49\cdot 52\cdot 57}{7!}h^8 \\ &+ \frac{76\cdot 79\cdot 84\cdot 91}{9!}h^{10} + \cdots\Bigg). \end{aligned}$$

In view of (2.6), the expansions above converge when

$$\left|\frac{e^{-3z/2}\sin(z\sqrt{3}/2)}{\sqrt{3}/2}\right| = \left|\frac{h}{\sqrt{3}}\right| \le \frac{e^{-\pi\sqrt{3}/6}}{\sqrt{3}}.$$

Since $h=\exp(-\pi(2n+1)\sqrt{3}/2)$, the aforementioned series converges for each n, $0\le n<\infty$.

2.4. The second main theorem in the quarterly reports is a beautiful generalization of Frullani's integral theorem. Employing Ramanujan's notation, let

$$f(x)-f(\infty)=\sum_{k=0}^{\infty}\frac{u(k)(-x)^k}{k!}\quad\text{and}\quad g(x)-g(\infty)=\sum_{k=0}^{\infty}\frac{v(k)(-x)^k}{k!},$$

where

$$f(\infty)=\lim_{x\to\infty}f(x)\quad\text{and}\quad g(\infty)=\lim_{x\to\infty}g(x),$$

which we assume exist.

Theorem II. *In the notation above, let f and g be continuous functions on* $[0, \infty)$. *Assume that* $u(s)/\Gamma(s+1)$ *and* $v(s)/\Gamma(s+1)$ *satisfy the hypotheses of Hardy's theorem. Furthermore, assume that* $f(0)=g(0)$ *and* $f(\infty)=g(\infty)$. *Then if* $a, b > 0$,

$$\lim_{n\to 0+} I_n \equiv \lim_{n\to 0+}\int_0^{\infty} x^{n-1}\{f(ax)-g(bx)\}\,dx$$

$$=\{f(0)-f(\infty)\}\left\{\text{Log}\left(\frac{b}{a}\right)+\frac{d}{ds}\left(\text{Log}\left(\frac{v(s)}{u(s)}\right)\right)_{s=0}\right\}. \tag{2.14}$$

Ramanujan tacitly assumes that the limit can be taken under the integral sign. If $f(x)=g(x)$, Theorem II reduces to

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\,dx=\{f(0)-f(\infty)\}\,\text{Log}\left(\frac{b}{a}\right), \tag{2.15}$$

since in this instance the limit on n can always be taken under the integral sign. Formula (2.15) is known as Frullani's theorem and holds for any continuous function f such that $f(\infty)$ exists. If $f(\infty)$ does not exist, but $f(x)/x$ is integrable over $[c, \infty)$ for some $c>0$, then (2.15) still holds, but with $f(\infty)$ replaced by 0. According to the reports, Ramanujan likely learned of Frullani's theorem from Williamson's book [1] on integral calculus. Ramanujan was also familiar with the classical text of J. Edwards [2, vol. 2, pp. 337–342] which has a nice section on Frullani's theorem and some generalizations.

Ramanujan's proof. Applying the Master Theorem with $0<n<1$, we find that

$$I_n=\int_0^{\infty}x^{n-1}(\{f(ax)-f(\infty)\}-\{g(bx)-g(\infty)\})\,dx$$

$$=\Gamma(n)\{a^{-n}u(-n)-b^{-n}v(-n)\}$$

$$=\Gamma(n+1)\left\{\frac{a^{-n}u(-n)-b^{-n}v(-n)}{n}\right\}.$$

Letting n tend to 0, we deduce that

$$\lim_{n\to 0} I_n = \lim_{n\to 0} \left\{\frac{b^n v(n) - a^n u(n)}{n}\right\}$$

$$= \lim_{n\to 0} \{b^n v(n) \operatorname{Log} b + b^n v'(n) - a^n u(n) \operatorname{Log} a - a^n u'(n)\}$$

$$= \{f(0) - f(\infty)\} \operatorname{Log}\left(\frac{b}{a}\right) + v'(0) - u'(0)$$

$$= \{f(0) - f(\infty)\} \left\{\operatorname{Log}\left(\frac{b}{a}\right) + \frac{d}{ds}\left(\operatorname{Log}\left(\frac{v(s)}{u(s)}\right)\right)_{s=0}\right\}, \qquad (2.16)$$

where we have used the fact that $u(0) = v(0) = f(0) - f(\infty)$.

Hardy [1], [19, pp. 195–226] discovered some different beautiful generalizations of Frullani's theorem. His paper contains a plethora of nice examples. In fact, Hardy evaluates ad hoc several integrals that fall under the province of Theorem II with $f \neq g$. Although Hardy was unsurpassed in the evaluation of integrals, he evidently failed to discover Theorem II. Another very interesting generalization of (2.15) was discovered by Lerch [1] and essentially rediscovered by Hardy [2], [3], [19, pp. 371–379]. The most complete source of information about Frullani's theorem is a paper of Ostrowski [2] which contains much historical information and several generalizations and ramifications. His paper [1] is a shorter, preliminary version of [2]. However, none of these papers contains Ramanujan's beautiful Theorem II, which evidently has not heretofore appeared in the literature.

2.5. This section is devoted to applications of Theorem II. Example (i) records Frullani's theorem. All of the relevant functions in the next three examples satisfy, by Stirling's formula, the hypotheses of Hardy's theorem. With each application of Theorem II, letting n tend to 0 under the integral sign is easily justified.

(ii) Recalling (1.5) and using the standard notation $\psi(x) = \Gamma'(x)/\Gamma(x)$, we find from Theorem II that for $a, b, p, q > 0$,

$$\int_0^\infty \frac{(1+ax)^{-p} - (1+bx)^{-q}}{x}\,dx = \operatorname{Log}\left(\frac{b}{a}\right) + \frac{d}{ds}\operatorname{Log}\left(\frac{\Gamma(q+s)\Gamma(p)}{\Gamma(p+s)\Gamma(q)}\right)_{s=0}$$

$$= \operatorname{Log}\left(\frac{b}{a}\right) + \psi(q) - \psi(p)$$

$$= \operatorname{Log}\left(\frac{b}{a}\right) + \sum_{k=0}^{\infty}\left\{\frac{1}{p+k} - \frac{1}{q+k}\right\}, \qquad (2.17)$$

since (see (5.5) of Chapter 6)

$$\psi(x) = -\gamma - \sum_{k=0}^{\infty} \left\{ \frac{1}{k+x} - \frac{1}{k+1} \right\}. \tag{2.18}$$

Ramanujan records four special cases of (2.17):

$$\int_0^\infty \frac{(1+ax)^{-5} - (1+bx)^{-7}}{x} dx = \operatorname{Log}\left(\frac{b}{a}\right) + \frac{11}{30},$$

$$\int_0^\infty \frac{(1+ax)^{-p} - (1+bx)^{-p-1}}{x} dx = \operatorname{Log}\left(\frac{b}{a}\right) + \frac{1}{p},$$

$$\int_0^\infty \frac{(1+ax)^{-p-1} - (1+bx)^{-p-5}}{x} dx = \operatorname{Log}\left(\frac{b}{a}\right) + \frac{2(2p+5)(p^2+5p+5)}{(p+1)(p+2)(p+3)(p+4)},$$

and

$$\int_0^\infty \frac{(1+ax)^{-1/4} - (1+bx)^{-3/4}}{x} dx = \operatorname{Log}\left(\frac{b}{a}\right) + \pi.$$

To calculate the last example employ Leibniz's series for $\pi/4$.

(iii) By (1.5), (1.12), Theorem II, and (2.18), for a, b, p, $q > 0$,

$$\begin{aligned}
\int_0^\infty \frac{(1+ax)^{-p} - \left(\dfrac{2}{1+\sqrt{1+4bx}}\right)^q}{x} dx
&= \operatorname{Log}\left(\frac{b}{a}\right) + \frac{d}{ds} \operatorname{Log}\left(\frac{q\Gamma(q+2s)\Gamma(p)}{\Gamma(q+s+1)\Gamma(p+s)}\right)_{s=0} \\
&= \operatorname{Log}\left(\frac{b}{a}\right) + 2\psi(q) - \psi(q+1) - \psi(p) \\
&= \operatorname{Log}\left(\frac{b}{a}\right) - \frac{1}{q} + \sum_{k=0}^{\infty} \left\{ \frac{1}{p+k} - \frac{1}{q+k} \right\}.
\end{aligned}$$

In particular,

$$\int_0^\infty \frac{(1+ax)^{-p} - \left(\dfrac{2}{1+\sqrt{1+4bx}}\right)^p}{x} dx = \operatorname{Log}\left(\frac{b}{a}\right) - \frac{1}{p}.$$

(iv) By (1.5), (1.13), Theorem II, and (2.18), for $a, b, p, q > 0$,

$$\int_0^\infty \frac{(1+ax)^{-p} - (bx + \sqrt{1+b^2x^2})^{-2q}}{x}\,dx$$

$$= \operatorname{Log}\left(\frac{b}{a}\right) + \frac{d}{ds}\left(\frac{q2^s\Gamma(q+\frac{1}{2}s)\Gamma(p)}{\Gamma(q+1-\frac{1}{2}s)\Gamma(p+s)}\right)_{s=0}$$

$$= \operatorname{Log}\left(\frac{b}{a}\right) + \operatorname{Log} 2 + \tfrac{1}{2}\psi(q) + \tfrac{1}{2}\psi(q+1) - \psi(p)$$

$$= \operatorname{Log}\left(\frac{2b}{a}\right) + \frac{1}{2q} + \sum_{k=0}^\infty \left\{\frac{1}{p+k} - \frac{1}{q+k}\right\}.$$

In particular,

$$\int_0^\infty \frac{(1+ax)^{-p} - (bx+\sqrt{1+b^2x^2})^{-2p}}{x}\,dx = \operatorname{Log}\left(\frac{2b}{a}\right) + \frac{1}{2p}.$$

Ramanujan remarks that "many other integrals such as

$$\int_0^\infty \frac{e^{-ax} - (1+bx)^{-p}}{x}\,dx, \quad \int_0^\infty \frac{e^{-ax}\cos^5(ax) - \left(\dfrac{2}{1+\sqrt{1+4bx}}\right)^p}{x}\,dx$$

can be found from Theorem II."

Let $f_1, \ldots, f_n$ be functions such that

$$\sum_{k=1}^n c_k f_k(0) = 0 = \sum_{k=1}^n c_k f_k(\infty),$$

where $c_1, \ldots, c_n$ are constants. Then it is clear from the proof of Theorem II that, under suitable conditions,

$$\int_0^\infty x^{-1} \sum_{k=1}^n c_k f(a_k x)\,dx$$

can be evaluated, where $a_1, \ldots, a_n > 0$.

James Hafner has kindly shown us that Theorem II can be significantly strengthened. Hafner's improvement allows Ramanujan's formula (2.14) to be applied to a wider variety of functions (and also for the limit in (2.14) to be taken under the integral sign). We now present Hafner's argument.

By Frullani's theorem (2.15),

$$\int_0^\infty \frac{f(ax)-g(bx)}{x}\,dx = \int_0^\infty \frac{\{f(ax)-f(bx)\} + \{f(bx)-g(bx)\}}{x}\,dx$$

$$= \{f(0)-f(\infty)\}\operatorname{Log}\left(\frac{b}{a}\right) + \int_0^\infty \frac{f(x)-g(x)}{x}\,dx.$$

By (2.16), it then remains to show that

$$\int_0^\infty \frac{f(x)-g(x)}{x}\,dx = -\{u'(0)-v'(0)\}.$$

Replacing $f(x)-g(x)$ by $f(x)$, it now suffices to prove the following lemma. (A similar result was established by Carlson [1] under stronger hypotheses.)

Lemma. *Suppose that f is analytic in a neighborhood of the nonnegative real axis Put*

$$f(z)=\sum_{k=0}^{\infty}\frac{u(k)(-z)^k}{k!},$$

for $|z|$ sufficiently small. Assume that, for some positive number δ,

$$\int^{\infty} f(x)x^{-\alpha}\,dx$$

converges uniformly for $1-\delta<\mathrm{Re}(\alpha)<2$. Suppose furthermore that $f(0)=0$. Then $u(s)$ can be extended in a neighborhood of $s=0$ so that u is differentiable at $s=0$ and so that

$$u'(0)=-\int_0^\infty \frac{f(x)}{x}\,dx. \tag{2.19}$$

Proof. For $\varepsilon>0$ sufficiently small, we shall, in fact, define u by

$$u(s)=\frac{\Gamma(s+1)e^{\pi i s}}{2\pi i}\oint_{|z|=\varepsilon} f(z)z^{-s-1}\,dz \tag{2.20}$$

and then show that u has the desired properties. (We shall assume that principal branches are always taken.) First, observe that if s is a nonnegative integer, (2.20) is valid by Cauchy's integral formula for derivatives. Since f is analytic in a neighborhood of the positive real axis, by Cauchy's theorem,

$$\int_{\varepsilon e^{2\pi i}}^{Re^{2\pi i}} f(z)z^{-s-1}\,dz-\int_{\varepsilon}^{R} f(z)z^{-s-1}\,dz=0, \qquad 0<\varepsilon<R<\infty.$$

It follows from (2.20) that

$$u(s)=\frac{\Gamma(s+1)e^{\pi i s}}{2\pi i}\left\{\oint_{|z|=\varepsilon} f(z)z^{-s-1}\,dz+(e^{-2\pi i s}-1)\int_{\varepsilon}^{R} f(x)x^{-s-1}\,dx\right\}.$$

Assuming that $-\delta<\mathrm{Re}(s)<1$ and using our hypotheses, we find that, upon letting ε tend to 0 and R tend to ∞,

$$\begin{aligned} u(s)&=\frac{\Gamma(s+1)(e^{-\pi i s}-e^{\pi i s})}{2\pi i}\int_0^\infty f(x)x^{-s-1}\,dx\\ &=-\frac{\Gamma(s+1)\sin(\pi s)}{\pi}\int_0^\infty f(x)x^{-s-1}\,dx.\end{aligned}$$

From this formula and our hypotheses, it follows that $u(s)$ is analytic in a neighborhood of $s = 0$. Calculating $u'(0)$ from the formula above, we deduce (2.19) to complete the proof.

Many of the integral evaluations found by Hardy [1] do not technically fall under the domain of Theorem II because the hypotheses are too restrictive. However, by using Hafner's lemma, the use of (2.14) can be justified. For example,

$$\int_0^\infty \left(\cos x - \frac{\sin x}{x}\right)\frac{dx}{x} = -1$$

and

$$\int_0^\infty \frac{\cos x - e^{-x^2}}{x}\,dx = -\frac{\gamma}{2},$$

where γ denotes Euler's constant.

2.6. Ramanujan now returns to his Master Theorem to derive several additional corollaries. As mentioned earlier, for brevity, we omit the hypotheses which arise from Hardy's theorem. We shall also write the integrands in terms of power series, as did Ramanujan, even though the series may not have an infinite radius of convergence.

Corollary (i). $\displaystyle\int_0^\infty x^{n-1} \sum_{k=0}^\infty \frac{\varphi(2k)(-x^2)^k}{(2k)!}\,dx = \Gamma(n)\varphi(-n)\cos(\tfrac{1}{2}\pi n).$

Ramanujan's proof. By the Master Theorem,

$$\int_0^\infty x^{n-1} \sum_{k=0}^\infty \frac{\varphi(2k)(-x)^k}{k!}\,dx = \Gamma(n)\varphi(-2n).$$

Replacing x by x^2 and n by $\frac{1}{2}n$, we find that

$$\int_0^\infty x^{n-1} \sum_{k=0}^\infty \frac{\varphi(2k)(-x^2)^k}{k!}\,dx = \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\varphi(-n).$$

Next replace $\varphi(s)$ by $\varphi(s)\Gamma(\frac{1}{2}s)/\Gamma(s)$ to obtain

$$\int_0^\infty x^{n-1} \sum_{k=0}^\infty \frac{\varphi(2k)(-x^2)^k}{(2k)!}\,dx = \frac{\Gamma(\frac{1}{2}n)\Gamma(-\frac{1}{2}n)\varphi(-n)}{4\Gamma(-n)},$$

from which formula (i) follows.

Corollary (ii) is simply (1.4) and is also found in Chapter 4, Section 11, Corollary 1.

Corollary (iii). $\displaystyle\int_0^\infty x^{n-1} \sum_{k=0}^\infty \varphi(2k)(-x^2)^k\,dx = \frac{\pi\varphi(-n)}{2\sin(\frac{1}{2}\pi n)}.$

Proof. Replace x by x^2, s by $\frac{1}{2}n$, and $\psi(s)$ by $\varphi(2s)$ in (1.4).

Corollary (iv). *If r is any natural number, then*

$$\int_0^\infty x^{n-1} \sum_{k=0}^\infty \varphi(rk)(-x^r)^k\,dx = \frac{\pi\varphi(-n)}{r\sin(\pi n/r)}.$$

Proof. The proof is similar to that of Corollary (iii), which is obviously a special case of Corollary (iv).

Corollary (v). $\displaystyle\int_0^\infty \sum_{k=0}^\infty \frac{\varphi(k)(-x)^k}{k!}\cos(nx)\,dx = \sum_{k=0}^\infty \varphi(-2k-1)(-n^2)^k.$

Ramanujan's proof. Expand $\cos(nx)$ into its Maclaurin series, invert the order of integration and summation of this series, and then apply the Master Theorem to each term.

For a rigorous proof of Corollary (v), see Hardy's paper [13], [21, pp. 280–289] or [20, pp. 200, 201]. Corollary (v) is also found in Chapter 4, Section 11, Corollary 3.

Corollary (vi).

$$\int_0^\infty \sum_{k=0}^\infty \frac{\varphi(k)(-x)^k}{k!} \sum_{k=0}^\infty \frac{\psi(2k)(-n^2x^2)^k}{(2k)!}\,dx = \sum_{k=0}^\infty \psi(2k)\varphi(-2k-1)(-n^2)^k.$$

Ramanujan's proof. His proof undoubtedly was like that of Corollary (v), which clearly is a special case of the present corollary.

Corollary (vi) is a corollary of formula (E), p. 186, and is discussed and proved on pages 202–205 of Hardy's book [20]. Corollary (vii) is Corollary 4 in Section 11 of Chapter 4.

Corollary (vii).

$$\int_0^\infty \sum_{k=0}^\infty \varphi(2k)(-x^2)^k\cos(nx)\,dx = \frac{\pi}{2}\sum_{k=0}^\infty \frac{\varphi(-k-1)(-n)^k}{k!}. \tag{2.21}$$

Ramanujan's proof. By the Cauchy multiplication of power series,

$$\int_0^\infty \sum_{k=0}^\infty \varphi(2k)(-x^2)^k\cos(nx)\,dx = \int_0^\infty \sum_{k=0}^\infty \varphi^*(2k)(-x^2)^k\,dx, \tag{2.22}$$

where

$$\varphi^*(k) = \sum_{j=0}^\infty \frac{\varphi(k-2j)n^{2j}}{(2j)!} - \sum_{j=1}^\infty \frac{\varphi(-2j)n^{k+2j}}{(k+2j)!}. \tag{2.23}$$

Note that (2.22) requires the definition of φ^* for only even k, but that

Ramanujan makes a definition for odd k as well. Now apply Corollary (iii) with $n = 1$ to the right side of (2.22). Using (2.23), we complete the proof of (2.21).

Ramanujan cautions us that if we expand $\cos(nx)$ into its Maclaurin series and apply Theorem I to each term, we will not obtain the correct formula in Corollary (vii). Moreover, if we replace $\varphi(s)$ by $\Gamma(s+1)\varphi(s)\cos(\pi s/2)$ in Corollary (v), we obtain the same formula as that gotten by expanding $\cos(nx)$ into its Maclaurin series! Ramanujan's advice is even more curious because he is criticizing the same procedure followed by him in establishing Corollary (v). We might therefore be suspicious of Ramanujan's new line of attack delineated above. Indeed, Ramanujan's argument seems to be objectionable for at least two reasons. First, there is no apparent reason why φ^*, defined by (2.23), should satisfy the hypotheses of Hardy's theorem, or even Theorem I. Secondly, the definition of φ^* itself invokes skepticism. The reader should now be unequivocally convinced that Corollary (vii) could not possibly be correct. But, despite our arguments to the contrary, Corollary (vii) does appear to be the correct formula! We shall defer our argument until after Corollary (viii) which generalizes Corollary (vii).

Corollary (viii).

$$\int_0^\infty \sum_{k=0}^\infty \varphi(2k)(-x^2)^k \sum_{k=0}^\infty \frac{\psi(2k)(-n^2x^2)^k}{(2k)!}\,dx = \frac{\pi}{2}\sum_{k=0}^\infty \frac{\varphi(-k-1)\psi(k)(-n)^k}{k!}.$$

Ramanujan's proof. Evidently, Ramanujan employed the same type of argument that he used to prove Corollary (vii), which is the special case $\psi(s) \equiv 1$ of Corollary (viii).

Proof. By Corollary (iii),

$$\int_0^\infty x^{s-1}\sum_{k=0}^\infty \varphi(2k)(-x^2)^k\,dx = \frac{\pi\varphi(-s)}{2\sin(\frac{1}{2}\pi s)}. \tag{2.24}$$

Letting $u = n^2x^2$ and employing the Master Theorem, we find that

$$\begin{aligned}\int_0^\infty x^{s-1} \sum_{k=0}^\infty \frac{\psi(2k)(-n^2x^2)^k}{(2k)!}\,dx &= \frac{n^{-s}}{2}\int_0^\infty u^{s/2-1}\sum_{k=0}^\infty \frac{\psi(2k)(-u)^k}{(2k)!}\,du \\ &= \frac{n^{-s}}{2}\int_0^\infty u^{s/2-1}\sum_{k=0}^\infty \frac{\psi(2k)\Gamma(k+1)(-u)^k}{\Gamma(2k+1)k!}\,du \\ &= \frac{n^{-s}\Gamma(\frac{1}{2}s)\psi(-s)\Gamma(1-\frac{1}{2}s)}{2\Gamma(1-s)} \\ &= n^{-s}\psi(-s)\cos(\tfrac{1}{2}\pi s)\Gamma(s),\end{aligned} \tag{2.25}$$

after a little simplification.

With the use of (2.24) and (2.25), we now apply Parseval's theorem for Mellin transforms (Titchmarsh [2, pp. 94, 95]) to deduce that, for $0 < c < 1$,

$$\int_0^\infty \sum_{k=0}^\infty \varphi(2k)(-x^2)^k \sum_{k=0}^\infty \frac{\psi(2k)(-n^2x^2)^k}{(2k)!}\,dx$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{2} n^{-1+s}\varphi(-s)\psi(s-1)\Gamma(1-s)\,ds$$
$$= \frac{\pi}{2}\sum_{k=0}^\infty n^k\varphi(-k-1)\psi(k)\frac{(-1)^k}{k!},$$

by an application of the residue theorem. This completes the proof.

We have obviously proceeded very formally above. Besides needing hypotheses to invoke Hardy's theorem, we also must require strong conditions on φ and ψ to apply Parseval's theorem. The last step of our proof also supposed that φ and ψ have sufficiently mild growth conditions so that we can integrate around a suitable rectangle with vertices of the form $c \pm iM$, $N \pm iM$ and let M and N tend to ∞ to get the required sum.

2.7. Ramanujan devotes Section 7 to the inversion formulas for Fourier cosine and Fourier sine transforms and some applications thereof. As we shall see, he derives these familiar formulas from his Master Theorem. The inversion formulae hold if φ is of bounded variation on every finite interval in $[0, \infty)$ and if φ is absolutely integrable over $(0, \infty)$ (Titchmarsh [1, pp. 434, 435], [2, pp. 16, 17]).

Theorem III. *Let* $n > 0$. *If*

$$\int_0^\infty \varphi(x)\cos(nx)\,dx = \psi(n), \tag{2.26}$$

then

$$\int_0^\infty \psi(x)\cos(nx)\,dx = \frac{\pi}{2}\varphi(n); \tag{2.27}$$

if

$$\int_0^\infty \varphi(x)\sin(nx)\,dx = \psi(n),$$

then

$$\int_0^\infty \psi(x)\sin(nx)\,dx = \frac{\pi}{2}\varphi(n).$$

Ramanujan's proof. If

$$\varphi(x) = \sum_{k=0}^\infty \frac{\varphi^*(k)(-x)^k}{k!},$$

then by Corollary (v) in the previous section, we observe that (2.26) holds with

$$\psi(x)=\sum_{k=0}^{\infty}\varphi^*(-2k-1)(-x^2)^k.$$

An application of Corollary (vii) then yields (2.27).

To prove the second part of Theorem III, Ramanujan evidently developed sine analogues of Corollaries (v) and (vii) and proceeded as above.

In the applications which follow, except for the last, the hypotheses of the aforementioned theorem in Titchmarsh's books [1], [2] are satisfied, and so the results are correct. In each example, $n>0$.

Example 1. Since

$$\int_0^{\infty} e^{-x}\cos(nx)\,dx=\frac{1}{1+n^2},$$

we have

$$\int_0^{\infty}\frac{\cos(nx)}{1+x^2}\,dx=\frac{\pi}{2}e^{-n}.$$

Example 2. Since (Gradshteyn and Ryzhik [1, p. 490])

$$\int_0^{\infty} x^{p-1}e^{-ax}\cos(nx)\,dx=\frac{\Gamma(p)\cos\{p\tan^{-1}(n/a)\}}{(a^2+n^2)^{p/2}},$$

where $a,\, p>0$, then

$$\Gamma(p)\int_0^{\infty}\frac{\cos\{p\tan^{-1}(x/a)\}}{(a^2+x^2)^{p/2}}\cos(nx)\,dx=\frac{\pi}{2}n^{p-1}e^{-an}.$$

Example 3 is the same as Example 2, except that all cosines are replaced by sines.

Example 4. First, recall that, for Re $a>0$ (Gradshteyn and Ryzhik [1, p. 480]),

$$\int_0^{\infty} e^{-ax^2}\cos(2nx)\,dx=\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-n^2/a}.$$

Setting Im $a=b$ and letting Re a tend to 0, we deduce with the aid of Abel's theorem (Bochner and Chandrasekharan [1, p. 35]) that

$$\int_0^{\infty}\cos(bx^2)\cos(2nx)\,dx=\frac{\sqrt{\pi}}{2\sqrt{b}}\cos\left(\frac{\pi}{4}-\frac{n^2}{b}\right)$$

and

$$\int_0^{\infty}\sin(bx^2)\cos(2nx)\,dx=\frac{\sqrt{\pi}}{2\sqrt{b}}\sin\left(\frac{\pi}{4}-\frac{n^2}{b}\right).$$

Replacing x, n, and b by $\sqrt{x}$, $\sqrt{a}$, and n, respectively, we find that the foregoing formulas become, respectively,

$$\int_0^\infty \frac{\cos(2\sqrt{ax})\cos(nx)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{n}}\cos\left(\frac{\pi}{4}-\frac{a}{n}\right)$$

and

$$\int_0^\infty \frac{\cos(2\sqrt{ax})\sin(nx)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{n}}\sin\left(\frac{\pi}{4}-\frac{a}{n}\right).$$

Ramanujan then invokes Theorem III to conclude that

$$\int_0^\infty \frac{\cos(\pi/4 - a/x)\cos(nx)}{\sqrt{x}}\,dx = \frac{\sqrt{\pi}}{2\sqrt{n}}\cos(2\sqrt{an})$$

and

$$\int_0^\infty \frac{\sin(\pi/4 - a/x)\sin(nx)}{\sqrt{x}}\,dx = \frac{\sqrt{\pi}}{2\sqrt{n}}\cos(2\sqrt{an}).$$

Note that $\cos(2\sqrt{ax})/\sqrt{x}$ is not absolutely integrable on $[0, \infty)$, and so the aforementioned theorem in Titchmarsh's text [1, pp. 434, 435] cannot be applied. Nonetheless, these last two evaluations are correct, as can be verified from results in Gradshteyn and Ryzhik's tables [1, pp. 398, 399]. (The formula at the top of p. 399 is incorrect; read $-e^{-2\sqrt{ab}}$ for $+e^{-2\sqrt{ab}}$.)

In 1952, Guinand [1] discovered that $\psi(x+1) - \operatorname{Log} x$ is self-reciprocal with respect to Fourier cosine transforms, where $\psi(x) = \Gamma'(x)/\Gamma(x)$. It is remarked in a footnote [1, p. 4] that this fact was independently discovered by T. A. Brown, who was informed by Hardy that this theorem is found in Ramanujan's quarterly reports. However, such a theorem is not found in the reports.

3. The Third Quarterly Report

3.1–3.3. After a sentence of introduction to the Board of Studies in Mathematics, Ramanujan returns to the conditions under which his Master Theorem is valid. He has become concerned about determining $\varphi(x)$ from its values on the set of nonnegative integers. Ramanujan tries to convince us that there is always a "natural" continuous function that is determined by its values at 0, 1, 2, ..., but his argument is quite unconvincing.

3.4–3.6. In these three sections, Ramanujan, for the first time in the reports, turns away from his Master Theorem and its ramifications. He discusses the ordinary composition of functions and extends this concept to "the fractional order of functions."

Define $F^0(x) = x$, $F^1(x) = F(x)$, and $F^n(x) = F(F^{n-1}(x))$, $n \geq 2$, for any function F, where, of course, the range of F^{n-1} must be contained in the domain of F. As a particular example, consider $F(x) = ax^p$, where $a, p, x > 0$. An easy inductive argument shows that

$$F^n(x) = a^{(p^n-1)/(p-1)}x^{p^n} \tag{3.1}$$

for each nonnegative integer n. Ramanujan now defines $F^n(x)$ for all real numbers n by (3.1). Thus, $F^{1/2}(x) = a^{1/(\sqrt{p}+1)}x^{\sqrt{p}}$. In general, if there exists a formula for $F^n(x)$ when n is a nonnegative integer, Ramanujan defines $F^n(x)$ for all real n by the same formula.

By inducting on m, it is easy to show that $F^m(F^n(x)) = F^{m+n}(x)$, where m and n are arbitrary nonnegative integers. Ramanujan declares that this identity remains valid for all real numbers m and n. For example, in the particular instance mentioned above, it is easily verified that $F^{1/2}(F^{1/2}(x)) = F(x)$.

Lemma. *Let $f = \chi^{-1}F\chi$. If n is a positive integer, then*

$$f^n = \chi^{-1}F^n\chi, \tag{3.2}$$

where χ^{-1} denotes the inverse function of χ.

This formula is easily established by induction on n. Ramanujan gives a more complicated version of (3.2) and assumes that (3.2) is valid for any real number n. He concludes Section 6 with several examples illustrating the Lemma's usefulness.

Example (a). Let $F(x) = \{a(x^m + 1)^p - 1\}^{1/m}$, where $a, p, x, m > 0$. Then, for any real number n,

$$F^n(x) = \{a^{(p^n-1)/(p-1)}(x^m + 1)^{p^n} - 1\}^{1/m}. \tag{3.3}$$

Proof. Let $\chi(x) = x^m + 1$. Then $\chi^{-1}(x) = (x-1)^{1/m}$. Letting $\varphi(x) = ax^p$, we find that $F = \chi^{-1}\varphi\chi$. By (3.2) and (3.1),

$$F^n(x) = \chi^{-1}\varphi^n\chi = \chi^{-1}\{a^{(p^n-1)/(p-1)}\{\chi(x)\}^{p^n}\},$$

from which (3.3) follows.

Example (b). Let $F(x) = x^2 - 2$, $x \geq 2$. Then

$$F^{1/2}(x) = \left(\frac{x + \sqrt{x^2-4}}{2}\right)^{\sqrt{2}} + \left(\frac{x - \sqrt{x^2-4}}{2}\right)^{\sqrt{2}},$$

$$F^{\text{Log } 3/\text{Log } 2}(x) = x^3 - 3x,$$

and

$$F^{\text{Log } 5/\text{Log } 2}(x) = x^5 - 5x^3 + 5x.$$

Proof. Set $x = y + 1/y$. Then $F(x) = y^2 + y^{-2}$, $F^2(x) = y^4 + y^{-4}$, and, in general, for $n \geq 0$,

$$F^n(x) = y^{2^n} + y^{-2^n} = \left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^{2^n} + \left(\frac{x - \sqrt{x^2 - 4}}{2}\right)^{2^n}. \tag{3.4}$$

The three proposed formulas now follow without difficulty.

Example (c), (i). If $F(x) = x^2 + 2x$, then for any real number n,

$$F^n(x) = (x + 1)^{2^n} - 1.$$

Proof. Let $\varphi(x) = x^2$ and $\chi(x) = x + 1$. Then $\chi^{-1}(x) = x - 1$. A short calculation shows that $F = \chi^{-1}\varphi\chi$. The proposed result now follows from (3.2).

Example (c), (ii). If $F(x) = x^2 - 2x$, $x \geq 3$, then for each real number n,

$$F^n(x) = 1 + \left(\frac{x - 1 + \sqrt{x^2 - 2x - 3}}{2}\right)^{2^n} + \left(\frac{x - 1 - \sqrt{x^2 - 2x - 3}}{2}\right)^{2^n}.$$

Proof. Let $\varphi(x) = x^2 - 2$ and $\chi(x) = x - 1$. Thus, $\chi^{-1}(x) = x + 1$ and $F = \chi^{-1}\varphi\chi$. Upon noting that (3.4) is valid with φ in place of F, we apply (3.2) to obtain the desired result.

Example (c), (iii). If $F(x) = x^2 + 4x$, $x \geq 0$, then for each real number n,

$$F^n(x) = \left\{\left(\frac{\sqrt{x + 4} + \sqrt{x}}{2}\right)^{2^n} - \left(\frac{\sqrt{x + 4} - \sqrt{x}}{2}\right)^{2^n}\right\}^2.$$

Proof. If $\varphi(x) = x^2 - 2$ and $\chi(x) = x + 2$, then $\chi^{-1}(x) = x - 2$ and $F = \chi^{-1}\varphi\chi$. By (3.2) and (3.4),

$$F^n(x) = \left(\frac{x + 2 + \sqrt{x^2 + 4x}}{2}\right)^{2^n} + \left(\frac{x + 2 - \sqrt{x^2 + 4x}}{2}\right)^{2^n} - 2.$$

After an elementary algebraic calculation, it is seen that the two formulas for $F^n(x)$ are in agreement.

Example (d). Let $0 < x \leq 1$. If

$$1 + \sqrt{F^{\operatorname{Log} 2}(x)} = \sqrt{\frac{1 - F^{\operatorname{Log} 2}(x)}{1 - x}}, \tag{3.5}$$

then

$$1 + 2\sqrt{\frac{F^{\operatorname{Log} 3}(x)}{x}} = \sqrt{\frac{1 - F^{\operatorname{Log} 3}(x)}{1 - x}}. \tag{3.6}$$

Proof. If we solve (3.5) for $F^{\operatorname{Log} 2}(x)$, we find that

$$G(x) \equiv F^{\operatorname{Log} 2}(x) = \left(\frac{x}{2 - x}\right)^2.$$

Let $\varphi(x) = x^2 - 2$ and $\chi(x) = 2/\sqrt{x}$. Then $\chi^{-1}(x) = 4/x^2$ and $\chi^{-1}\varphi\chi = G$. By (3.4),

$$\varphi^n(2/\sqrt{x}) = \left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)^{2^n} + \left(\frac{1-\sqrt{1-x}}{\sqrt{x}}\right)^{2^n}.$$

Thus, by (3.2),

$$F^{n \operatorname{Log} 2}(x) = G^n(x) = \frac{4}{\left\{\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)^{2^n} + \left(\frac{1-\sqrt{1-x}}{\sqrt{x}}\right)^{2^n}\right\}^2}.$$

In particular,

$$F^{\operatorname{Log} 3}(x) = \frac{4}{\left\{\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)^{3} + \left(\frac{1-\sqrt{1-x}}{\sqrt{x}}\right)^{3}\right\}^2} = \frac{x^3}{(4-3x)^2}.$$

A straightforward calculation now shows that $F^{\operatorname{Log} 3}(x)$ satisfies (3.6), and so the proof is complete.

Although the arguments in these three sections have been purely formal, the fractional iteration of functions can be put on a firm basis; see Comtet's book [1, pp. 144–148].

3.7–3.9. In Sections 7–9, Ramanujan, inspired by reading Hardy's tract [6], briefly studies orders of infinity. None of Ramanujan's results is new, but his approach seems novel.

As in Hardy's book [6],

$$f(x) \prec g(x)$$

means that $\lim_{x\to\infty} f(x)/g(x) = 0$, and we say that $g(x)$ is of an order higher than $f(x)$, or $f(x)$ is of an order lower than $g(x)$. The symbol $\succ$ is defined similarly. Thus,

$$x \prec x^2 \prec x^3 \prec \cdots$$

and

$$e^{x^2} \succ x^x \succ \Gamma(x+1) \succ e^x \succ x^n,$$

for every real number n.

Iterative powers may be defined inductively from the definition $a^{b^c} = a^{(b^c)}$. Let ${}^n e^x$ denote the nth iterate of the exponential function. Thus, ${}^1 e^x = e^x$, ${}^2 e^x = e^{e^x}$, etc. We also shall write ${}^n e^1 = {}^n e$. From the ratio test, it can be seen that

$$f(x) = 1 + \frac{e^x}{2^3} + \frac{{}^2 e^x}{2^{3^4}} + \frac{{}^3 e^x}{2^{3^{4^5}}} + \cdots$$

converges for every x. Furthermore, $f(x) \succ {}^n e^x$ for each positive integer n.

On the other hand, by a clever construction, Ramanujan next shows that there exist functions tending to ∞ slower than any iterate of the logarithmic function. (This is actually a consequence of a theorem of du Bois–Reymond found in Hardy's tract [6, p. 8].) Define $\operatorname{Log}_1 x = \operatorname{Log} x$ and $\operatorname{Log}_n x = \operatorname{Log}(\operatorname{Log}_{n-1} x)$, $n \geq 2$.

Theorem IV. *There exists a function f such that $f(x) \prec \operatorname{Log}_n x$ for each positive integer n.*

Proof. Let $\varphi(x)$ be any positive, continuous function such that $\varphi(0) = 1$ and $x\varphi(x) = \varphi(\operatorname{Log} x)$, $x > 0$. Define $u_0 = \varphi(1) = 1$ and $u_n = \varphi({}^n e)$, $n \geq 1$. Let

$$\int_0^1 \varphi(x)\, dx = C.$$

From the definition of φ it follows that

$$\int_1^e \varphi(x)\, dx = \int_1^e \frac{\varphi(\operatorname{Log} x)}{x}\, dx = \int_0^1 \varphi(x)\, dx = C,$$

$$\int_e^{e^e} \varphi(x)\, dx = \int_e^{e^e} \frac{\varphi(\operatorname{Log} x)}{x}\, dx = \int_1^e \varphi(x)\, dx = C,$$

and, in general, for $n \geqslant 2$,

$$\int_{{}^n e}^{{}^{n+1} e} \varphi(x)\, dx = \int_{{}^n e}^{{}^{n+1} e} \frac{\varphi(\operatorname{Log} x)}{x}\, dx = \int_{{}^{n-1} e}^{{}^n e} \varphi(x)\, dx = C, \tag{3.7}$$

by induction. Thus,

$$f(x) \equiv \int_0^x \varphi(t)\, dt$$

tends to ∞ as x tends to ∞. By L'Hôpital's rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{\operatorname{Log}_n x} &= \lim_{x\to\infty} \varphi(x) x \operatorname{Log} x \operatorname{Log}_2 x \cdots \operatorname{Log}_{n-1} x \\
&= \lim_{x\to\infty} \varphi(\operatorname{Log} x) \operatorname{Log} x \operatorname{Log}_2 x \cdots \operatorname{Log}_{n-1} x \\
&= \lim_{x\to\infty} \varphi(\operatorname{Log}_2 x) \operatorname{Log}_2 x \cdots \operatorname{Log}_{n-1} x \\
&= \cdots = \lim_{x\to\infty} \varphi(\operatorname{Log}_n x) = 0,
\end{aligned}$$

i.e., $f(x) \prec \operatorname{Log}_n x$ for every positive integer n.

Since $f(x)$ tends to ∞ as x tends to ∞, the series

$$\Phi \equiv \sum_{k=1}^{\infty} \varphi(k)$$

diverges by the integral test. Ramanujan claims that Φ diverges so slowly that the sum of $10^{2000000}$ terms does not exceed 5. This claim is meaningless unless C is specified. Note that $n=5$ is the least positive integer such that ${}^{n}e > 10^{2000000}$. Then by a familiar geometrical argument from calculus,

$$\sum_{k=1}^{10^{2000000}} \varphi(k) < 1 + \int_1^{10^{2000000}} \varphi(x)\,dx < 1 + \int_1^{{}^5e} \varphi(x)\,dx = 1 + 5C,$$

by (3.7). Thus, if $C \leq \frac{4}{5}$, Ramanujan's claim is justified.

To further illustrate slowly divergent series, Ramanujan remarks that

$$\sum_{k=2}^{10^{29}} \frac{1}{k \operatorname{Log} k} < 5.$$

Indeed, Hardy [6, p. 61] and Boas [1, p. 244], [2, p. 156] have shown that 1.3×10^{29} terms are required to exceed 5.

In Section 9, Ramanujan discusses "scales of infinity." Thus, x, $x^{1/2}$, $x^{1/3}$, and $x^{1/4}$ belong to the "ordinary scale," and $\operatorname{Log} x$, $(\operatorname{Log} x)^{1/3}$, and $\operatorname{Log}_2 x$ are in the "logarithmic scale." If f is any member of one scale and g is any member of another distinct scale, then $f \prec g$ always or $g \prec f$ always. Using an argument similar to the proof of Theorem IV, Ramanujan attempts to show that the number of scales is infinite, which again is a consequence of du Bois–Reymond's theorem. However, Ramanujan's attempted proof is flawed in several respects.

3.10. Ramanujan now returns to the topic of fractional composition of functions. His main task is to expand $f^r(x)$ as a power series in r. As before, Ramanujan's procedure is not rigorous.

Some of this material is found at the beginning of Chapter 4. Thus, the first part of our description here will be rather brief.

Write

$$f^r(x) = \sum_{k=0}^{\infty} \frac{\psi_k(x) r^k}{k!}. \tag{3.8}$$

Putting $r=0$, we find that $x = f^0(x) = \psi_0(x)$. Ramanujan next shows that

$$\frac{df^r(x)}{dr} = \psi_1(x) \frac{df^r(x)}{dx} \tag{3.9}$$

from which it can easily be deduced that $\psi_n(x) = \psi_1(x)\, d\psi_{n-1}(x)/dx$, $n \geq 1$. (See the proof of Entry 4 in Chapter 4.) Thus, in order to calculate the expansion (3.8), we need only to determine $\psi_1(x)$.

Ramanujan's determination of $\psi_1(x)$ is quite interesting but not rigorous. Let $F(x)$ be a solution of

$$F(x) = f'(x) F\{f(x)\}, \tag{3.10}$$

and let

$$C(x) = \int_x^{f(x)} F(t)\,dt.$$

By differentiating $C(x)$ and employing (3.10), we find that $C(x)$ is independent of x, and so we write $C(x) = C$. Now for each integer $n \geq 1$,

$$\int_{f^{n-1}(x)}^{f^n(x)} F(t)\,dt = \int_{f^{n-1}(x)}^{f^n(x)} f'(t)F\{f(t)\}\,dt = \int_{f^n(x)}^{f^{n+1}(x)} F(t)\,dt.$$

Hence, for each positive integer r,

$$\int_x^{f^r(x)} F(t)\,dt = rC. \tag{3.11}$$

But now Ramanujan differentiates (3.11) *with respect to r*. Using (3.9), he finds that

$$\psi_1(x)\frac{df^r(x)}{dx}F\{f^r(x)\} = C. \tag{3.12}$$

By a repeated use of (3.10) along with the chain rule, Ramanujan next finds that, for each positive integer r,

$$F(x) = \frac{df^r(x)}{dx}F\{f^r(x)\}. \tag{3.13}$$

Comparing (3.12) and (3.13), we deduce that $\psi_1(x) = C/F(x)$. As previously shown, this enables Ramanujan to determine the series (3.8).

3.11. As a natural outgrowth of his study of the fractional composition of functions, Ramanujan briefly studies fractional differentiation in Section 11. For each nonnegative integer n, let $D^n f(x) = f^{(n)}(x)$. Ramanujan assumes that there exists a unique, "natural" function of n passing through the points $D^0 f(x)$, $Df(x)$, $D^2 f(x), \ldots$, and in this rather imprecise fashion defines fractional derivatives of f. As an example, consider $f(x) = e^{ax}$. For each nonnegative integer n,

$$D^n e^{ax} = a^n e^{ax}, \tag{3.14}$$

and so it is natural to define the nth derivative of e^{ax} for any real number n by (3.14). It is interesting that Liouville in 1832 began his study of fractional differentiation in this same manner. (See an article by Ross [1].)

Theorem V. *If* $n > 0$, *then*

$$\int_0^\infty x^{n-1} f^{(r)}(a-x)\,dx = \Gamma(n) f^{(r-n)}(a), \tag{3.15}$$

where $f^{(k)}(t)$ *denotes the* kth *fractional derivative of* f.

Ramanujan points out that Theorem V can actually be used to *define* the fractional derivative $f^{(k)}$. Let r be any nonnegative integer greater than k and let $n = r - k$. Thus; $n > 0$, and by (3.15),

$$\int_0^\infty x^{n-1} f^{(r)}(a - x)\,dx = \Gamma(n) f^{(k)}(a). \tag{3.16}$$

Since the left side of (3.16) has a definite meaning from elementary calculus, (3.16) can be used to define the fractional derivative $f^{(k)}(a)$ for any real number k. It is remarkable that (3.16) is precisely the same definition that Liouville gave for the fractional derivative $f^{(k)}(a)$ (Ross [1, p. 6, equation (7)]).

Ramanujan's deduction of (3.15) is purely formal and amazingly simple. As we shall see below, he deduces (3.15) from his Master Theorem!

Ramanujan's proof. By Taylor's theorem and the Master Theorem,

$$\int_0^\infty x^{n-1} f^{(r)}(a - x)\,dx = \int_0^\infty x^{n-1} \sum_{k=0}^\infty \frac{f^{(r+k)}(a)(-x)^k}{k!}\,dx$$
$$= \Gamma(n) f^{(r-n)}(a).$$

Corollary. *We have*

$$\int_0^\infty f^{(r)}(a - x^2)\,dx = \frac{\sqrt{\pi}}{2} f^{(r-1/2)}(a).$$

Proof. Set $n = \frac{1}{2}$ and replace x by x^2 in (3.15).

For an introduction to fractional calculus, see the book by Oldham and Spanier [1]. An informative historical account of fractional calculus and many of its applications can be found in a book edited by Ross [1]; for another historical account, see Ross's paper [2].

3.12. In Section 12, Ramanujan indicates how operators can be formally employed to deduce various types of formulas. As at the beginning of the third report, Ramanujan incorrectly asserts that a function can be uniquely determined from its values on the set of nonnegative integers in order to justify his methods.

As is customary, the difference operator Δ^n, $n \geq 1$, is defined as follows. Define $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^n f(x) = \Delta(\Delta^{n-1} f(x))$, $n \geq 2$. Let $E = I + \Delta$, where I is the identity operator. Then $E^n \varphi(0) = \varphi(n)$ for any nonnegative integer n. Ramanujan assumes that the last equality is valid for any real number n and deduces that

$$\varphi(n) = E^n \varphi(0) = (I + \Delta)^n \varphi(0) = \sum_{k=0}^\infty \binom{n}{k} \Delta^k \varphi(0),$$

which can be thought of as an infinite version of Newton's interpolation formula (Abramowitz and Stegun [1, p. 880]). We make no attempt to justify this formal process.

Ramanujan prefaces his next application by declaring that, "If a result is true *only* for real values of a quantity (say a), then the result got by using the operator for a is true only if when the new function can be expressed in terms of the original function ...." As an illustration, Ramanujan considers

$$\int_0^\infty \frac{\cos(ax)}{x^2+1}\,dx = \frac{\pi}{2}e^{-a},$$

which is valid only for (nonnegative) real values of a. Multiplying both sides by $\varphi(0)$ and replacing a by the differential operator D, we get

$$\int_0^\infty \frac{\cos(Dx)}{x^2+1}\varphi(0)\,dx = \frac{\pi}{2}e^{-D}\varphi(0). \tag{3.17}$$

Now

$$\begin{aligned}\cos(Dx)\varphi(0) &= \frac{e^{iDx}+e^{-iDx}}{2}\varphi(0)\\ &= \tfrac{1}{2}\sum_{k=0}^{\infty}\frac{(iDx)^k+(-iDx)^k}{k!}\varphi(0)\\ &= \tfrac{1}{2}\sum_{k=0}^{\infty}\frac{(ix)^k+(-ix)^k}{k!}\varphi^{(k)}(0) = \frac{\varphi(ix)+\varphi(-ix)}{2}.\end{aligned}$$

Similarly, $e^{-D}\varphi(0)=\varphi(-1)$. Thus, (3.17) can be written in the form

$$\int_0^\infty \frac{\varphi(ix)+\varphi(-ix)}{2(x^2+1)}\,dx = \frac{\pi}{2}\varphi(-1), \tag{3.18}$$

which Ramanujan claims is valid only when $\varphi(ix)+\varphi(-ix)$ can be written as a linear combination of cosines. Observe that, in fact, (3.18) is really a special instance of Corollary (viii) in the second report.

Ramanujan continues the remarks quoted above by asserting that, "But if a result is true for complex values of a then we can freely use the operators ...." To elucidate this principle, consider

$$\int_0^\infty x^{n-1}e^{-ax}\,dx = a^{-n}\Gamma(n),$$

which is valid when $\mathrm{Re}(n), \mathrm{Re}(a) > 0$. Multiply both side by $\varphi(0)$ and replace a by the operator E. Since $E^r\varphi(0)=\varphi(r)$,

$$\begin{aligned}&\int_0^\infty x^{n-1}\sum_{k=0}^{\infty}\frac{(-Ex)^k\varphi(0)}{k!}\,dx\\ &\quad = \int_0^\infty x^{n-1}\sum_{k=0}^{\infty}\frac{\varphi(k)(-x)^k}{k!}\,dx = \Gamma(n)E^{-n}\varphi(0) = \Gamma(n)\varphi(-n),\end{aligned}$$

and so another "proof" of the Master Theorem is obtained.

Ramanujan applies operators in two more instances to rederive the result of Example (iv) in the first report and Theorem III in the second report.

3.13. Section 13 is devoted to the delineation of some results that are described more fully in Ramanujan's paper [13], [15, pp. 53–58]. First [15, p. 54],

$$\int_0^\infty |\Gamma(a+ix)|^2 \cos(2nx)\,dx = \tfrac{1}{2}\sqrt{\pi}\,\Gamma(a)\Gamma(a+\tfrac{1}{2})\,\mathrm{sech}^{2a} n,$$

where $a > 0$. By the Fourier cosine inversion formula of Theorem III,

$$\int_0^\infty \frac{\cos(2nx)}{\cosh^{2a} x}\,dx = \frac{\sqrt{\pi}\,|\Gamma(a+in)|^2}{2\Gamma(a)\Gamma(a+\tfrac{1}{2})}. \tag{3.19}$$

Letting $a = \frac{5}{2}$ and using the fact that $|\Gamma(\frac{1}{2}+iy)|^2 = \pi\,\mathrm{sech}(\pi y)$ (Gradshteyn and Ryzhik [1, p. 937]), we find from (3.19) that

$$\int_0^\infty \frac{\cos(nx)}{\cosh^5 x}\,dx = \frac{\pi(n^2+1)(n^2+9)}{48\cosh(\pi n/2)}.$$

Replacing a by $1-a$ in (3.19) and using an infinite product representation for $\cosh(2\pi n) - \cos(2\pi a)$ (Gradshteyn and Ryzhik [1, p. 37]), we find after some manipulation and simplification that, for $0 < a < 1$,

$$\int_0^\infty \frac{\cos(2nx)}{\cosh^{2a} x}\,dx \int_0^\infty \frac{\cos(2nx)}{\cosh^{2(1-a)} x}\,dx = \frac{\pi \sin(2\pi a)}{2(1-2a)\{\cosh(2\pi n) - \cos(2\pi a)\}}.$$

3.14. Let $a, h, n > 0$. First,

$$\int_0^h e^{-ax}\cos(nx)\,dx = \mathrm{Re}\int_0^h e^{-(a+ni)x}\,dx$$
$$= \frac{a\{1 - e^{-ah}\cos(hn)\} + ne^{-ah}\sin(hn)}{a^2+n^2}.$$

Secondly, using some integral evaluations in Gradshteyn and Ryzhik's tables [1, p. 406, Section 3.723, formulas 2, 3], we find that

$$\int_0^\infty \frac{a\{1 - e^{-ah}\cos(hx)\} + xe^{-ah}\sin(hx)}{a^2+x^2}\cos(nx)\,dx$$
$$= \int_0^\infty \frac{\cos(nx)}{a^2+x^2}\,dx - \frac{ae^{-ah}}{2}\int_0^\infty \frac{\cos\{(h+n)x\}}{a^2+x^2}\,dx$$
$$- \frac{ae^{-ah}}{2}\int_0^\infty \frac{\cos\{(h-n)x\}}{a^2+x^2}\,dx$$
$$+ \frac{e^{-ah}}{2}\int_0^\infty \frac{x\sin\{(h+n)x\}}{a^2+x^2}\,dx$$
$$+ \frac{e^{-ah}}{2}\int_0^\infty \frac{x\sin\{(h-n)x\}}{a^2+x^2}\,dx$$

$$= \frac{\pi}{2} e^{-an} - \frac{\pi}{4} e^{-ah-a|h-n|} + \operatorname{sgn}(h-n) \frac{\pi}{4} e^{-ah-a|h-n|}$$

$$= \begin{cases} (\pi/2)e^{-an}, & \text{if } n < h, \\ (\pi/4)e^{-an}. & \text{if } n = h, \\ 0, & \text{if } n > h. \end{cases}$$

Using differential operators in a manner like that in (3.17) and employing the calculations above and similar calculations with $\cos(nx)$ replaced by $\sin(nx)$, Ramanujan deduces the following theorem.

Theorem VI. *Let* $n > 0$. *If*

$$\psi(n) = \int_0^h \varphi(x) \cos(nx)\, dx,$$

then

$$\int_0^\infty \psi(x) \cos(nx)\, dx = \begin{cases} (\pi/2)\varphi(n), & \text{if } n < h, \\ (\pi/4)\varphi(n), & \text{if } n = h, \\ 0, & \text{if } n > h; \end{cases}$$

if

$$\psi(n) = \int_0^h \varphi(x) \sin(nx)\, dx,$$

then

$$\int_0^\infty \psi(x) \sin(nx)\, dx = \begin{cases} (\pi/2)\varphi(n), & \text{if } n < h, \\ (\pi/4)\varphi(n), & \text{if } n = h, \\ 0, & \text{if } n > h. \end{cases}$$

Now, in fact, Theorem VI follows from the Fourier integral theorem (Titchmarsh [2, pp. 16, 17], [1, pp. 432–435]) and is valid when φ is continuous and of bounded variation on $[0, h]$.

3.15. Several of the formulas in this final section have been discussed by Hardy [20, pp. 187, 206–209] who regarded them as examples of Ramanujan's formalistic thinking. Like Hardy, we shall make no attempt to justify the formal analysis.

Theorem VII. *Suppose that* F *and* f *satisfy either*

$$\text{(a)} \qquad \int_0^\infty F(ax) f(bx)\, dx = \frac{1}{a+b}, \qquad a, b > 0,$$

or

$$\text{(b)} \qquad \int_0^\infty x^{p-1}F(x)\,dx \int_0^\infty x^{-p}f(x)\,dx = \frac{\pi}{\sin(\pi p)}, \qquad 0<p<1.$$

If

$$\text{(c)} \qquad \int_0^\infty \varphi(x)\tfrac{1}{2}\{F(nxi)+F(-nxi)\}\,dx = \psi(n), \qquad n>0,$$

then

$$\text{(d)} \qquad \int_0^\infty \psi(x)\tfrac{1}{2}\{f(nxi)+f(-nxi)\}\,dx = \frac{\pi}{2}\varphi(n), \qquad n>0.$$

Ramanujan's proof. Ramanujan's argument is not logically consistent with the formulation of his theorem. He really proceeds as follows.

Set

$$\varphi(x) = \sum_{k=0}^\infty \frac{\varphi^*(k)(-x)^k}{k!}, \qquad \psi(x) = \sum_{k=0}^\infty \psi^*(2k)(-x^2)^k,$$

$$F(x) = \sum_{k=0}^\infty \frac{F^*(k)(-x)^k}{k!}, \quad \text{and} \quad f(x) = \sum_{k=0}^\infty \frac{f^*(k)(-x)^k}{k!}.$$

Suppose that $\psi^*(r) = F^*(r)\varphi^*(-r-1)$ and $f^*(r)F^*(-r-1) = 1$. Then he *proves* (a), (b), (c), and (d) in turn.

Formulas (c) and (d) are then just reformulations of Corollaries (vi) and (viii), respectively, in Section 2.6. Hardy [20, p. 206] has related Ramanujan's formal proof of (a), and so we omit it. To obtain (b), simply apply Ramanujan's Master Theorem to each of the two integrals and then use the hypothesis $F^*(-r)f^*(r-1) = 1$.

The third quarterly report concludes with five corollaries of Theorem VII.

For the first application, let $F(x) = f(x) = e^{-x}$. Then

$$\int_0^\infty e^{-ax}e^{-bx}\,dx = \frac{1}{a+b}, \qquad a,\ b>0.$$

Thus, Theorem VII yields a restatement of the Fourier cosine inversion formula of Theorem III.

Corollary 2. *Let* $n>0$. *If*

$$\int_0^\infty \frac{\varphi(x)}{1-n^2x^2}\,dx = \psi(n),$$

then

$$\int_0^\infty \frac{\psi(x)}{1-n^2x^2}\,dx = \frac{\pi^2}{4}\varphi(n).$$

Ramanujan's proof. Let $F(x) = 1/(1+x^2)$ and $f(x) = 2F(x)/\pi$. Then

$$\frac{2}{\pi}\int_0^\infty \frac{dx}{(1+a^2x^2)(1+b^2x^2)} = \frac{1}{a+b}, \qquad a, b > 0.$$

Applying Theorem VII, we readily deduce the desired result.

For conditions under which Corollary 2 is valid, see Titchmarsh's book [2, Chapter 8, esp. p. 219, paragraph (10)].

The next corollary gives the inversion formula for Hankel transforms. Consult Titchmarsh's treatise [2, pp. 240–242] for conditions which insure the validity of Corollary 3 and for a proof. Even if v is an integer, it does not appear that Corollary 3 can be derived from Theorem VII. Ramanujan is quite vague on the origin of Corollary 3.

Corollary 3. *Let $n > 0$. If*

$$\int_0^\infty x\varphi(x)J_v(nx)\,dx = \psi(n),$$

then

$$\int_0^\infty x\psi(x)J_v(nx)\,dx = \varphi(n),$$

where J_v denotes the ordinary Bessel function of order v.

Corollary 4. *If $\alpha/m = (n-\beta)/n = p$, where $m, n > 0$ and $0 < p < 1$, and*

$$\int_0^\infty F(ax)f(bx)\,dx = \frac{1}{a+b}, \qquad a, b > 0,$$

then

$$\int_0^\infty x^{\alpha-1}F(x^m)\,dx \int_0^\infty x^{\beta-1}f(x^n)\,dx = \frac{\pi}{mn\sin(\pi p)}.$$

Corollary 4 is easily derived from formula (b) of Theorem VII. (Again, we emphasize that Theorem VII is illogically formulated.) Corollary 5 is the special instance of Corollary 4 when $m = n = 2$, $\alpha = \beta = 1$, and $p = \frac{1}{2}$.

Corollary 5. *If*

$$\int_0^\infty F(ax)f(bx)\,dx = \frac{1}{a+b},$$

then

$$\int_0^\infty F(x^2)\,dx \int_0^\infty f(x^2)\,dx = \frac{\pi}{4}.$$

Conclusion

Although Ramanujan did not fully develop many of the ideas in his quarterly reports, he discovered the basic underlying formulas for several theories, most of which were developed earlier but some of which evolved later. It is rather remarkable that Ramanujan's formulas are almost invariably correct, even though his methods were generally without a sound theoretical foundation. His amazing insights enabled him to determine when his formal arguments led to bona fide formulas and when they did not. Perhaps Ramanujan's work contains a message for contemporary mathematicians. We might allow our untamed, formal arguments more freedom to roam without worrying about how to return home in order to find new paths to the other side of the mountain.

References

Abel, N. H.

[1] Démonstration d'une expression de laquelle la formule binome est un cas particulier, *J. Reine Angew. Math.*, **1** (1826), 159–160.

[2] *Oeuvres*, vol. 1, Grøndahl & Son, Christiania, 1881.

[3] *Oeuvres*, vol. 2, Grøndahl & Son, Christiania, 1881.

Abramowitz, M. and Stegun, I. A., editors

[1] *Handbook of Mathematical Functions*, Dover, New York, 1965.

Acreman, D. and Loxton, J. H.

[1] Asymptotic analysis of Ramanujan pairs, *Aequa. Math.*, to appear.

Aitken, A. C.

[1] On Bernoulli's numerical solution of algebraic equations, *Proc. Royal Soc. Edinburgh*, Ser. A, **46** (1926), 289–305.

Aiyar, P. V. Seshu

[1] The late Mr. S. Ramanujan, *J. Indian Math. Soc.*, **12** (1920), 81–86.

Andrews, G. E.

[1] *The Theory of Partitions*, Addison-Wesley, Reading, Massachusetts, 1976.

[2] An introduction to Ramanujan's "lost" notebook, *Amer. Math. Monthly*, **86** (1979), 89–108.

[3] An incredible formula of Ramanujan, *Austral. Math. Soc. Gaz.*, **6** (1979), 80–89.

Andrews, J. J. and Lacher, R. C.

[1] Stacked exponents, *Aequa. Math.*, **16** (1977), 137–147.

Andrews, W. S.

[1] *Magic Squares and Cubes*, 2nd ed., Dover, New York, 1960.

Apéry, R.

[1] Irrationalité de $\zeta(2)$ et $\zeta(3)$, *Journées arithmétiques Luminy, Astérisque*, **61** (1979), 11–13.

Apostol, T. M.

[1] *Mathematical Analysis*, 1st ed., Addison-Wesley, Reading, Massachusetts, 1957.

Apostol, T. M. and Vu, T. H.
[1] Dirichlet series related to the Riemann zeta-function, *J. Number Theory*, **19** (1984), 85–102.

Appell, P.
[1] Sur une classe de fonctions analogues aux fonctions Eulériennes, *Math. Ann.*, **19** (1882), 84–102.

Askey, R.
[1] Ramanujan's extensions of the gamma and beta functions, *Amer. Math. Monthly*, **87** (1980), 346–359.

Ayoub, R.
[1] *An Introduction to the Analytic Theory of Numbers*, American Mathematical Society, Providence, 1963.
[2] Euler and the zeta-function, *Amer. Math. Monthly*, **81** (1974), 1067–1086.

Baker, I. N.
[1] The iteration of entire transcendental functions and the solution of the functional equation $f\{f(z)\} = F(z)$, *Math. Ann.*, **129** (1955), 174–180.
[2] Zusammensetzungen ganzer Funktionen, *Math. Z.*, **69** (1958), 121–163.

Balakrishnan, U.
[1] On the Laurent expansion of $\zeta(s, a)$ at $s = 1$, *J. Indian Math. Soc.*, **46** (1982).
[2] Laurent expansion of Dirichlet series, to appear.

Balasubrahmanian, N.
[1] On the number defined by $N_r = (1/e)\sum_{n=0}^{\infty}(n^r/n!)$, *Math. Student*, **18** (1950), 130–132.

Barnes, E. W.
[1] The theory of the gamma function, *Mess. Math.*, **29** (1900), 64–128.
[2] Theory of the double gamma function, *Phil. Trans. Royal Soc. London* A, **196** (1901), 265–387.
[3] The generalisation of the Maclaurin sum formula, and the range of its applicability, *Quart. J. Math.*, **35** (1904), 175–188.

Barrow, D. W.
[1] Infinite exponentials, *Amer. Math. Monthly*, **43** (1936), 150–160.

Becker, H. W. and Riordan, J.
[1] The arithmetic of Bell and Stirling numbers, *Amer. J. Math.*, **70** (1948), 385–391.

Bell, E. T.
[1] Exponential polynomials, *Ann. Math.*, **35** (1934), 258–277.
[2] The iterated exponential integers, *Ann. Math.*, **39** (1938), 539–557.

Bender, C. M. and Orszag, S. A.
[1] *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, New York, 1978.

Bendersky, L.
[1] Sur la fonction gamma généralisée, *Acta Math.*, **61** (1933), 263–322.

Berndt, B. C.
[1] On the Hurwitz zeta-function, *Rocky Mountain J. Math.*, **2** (1972), 151–157.
[2] Elementary evaluation of $\zeta(2n)$, *Math. Mag.*, **48** (1975), 148–154.
[3] Ramanujan's notebooks, *Math. Mag.*, **51** (1978), 147–164.
[4] The gamma function and the Hurwitz zeta-function, *Amer. Math. Monthly*, **91** (1985), 126–130.

Berndt, B. C. and Schoenfeld, L.
[1] Periodic analogues of the Euler–Maclaurin and Poisson summation formulas with applications to number theory, *Acta Arith.*, **28** (1975), 23–68.

Bharathi, R., editor
[1] *Prof. Srinivasa Ramanujan Commemoration Volume*, Jupiter Press, Madras, 1974.

Birkeland, R.
[1] Über die Auflösung algebraischer Gleichungen durch hypergeometrische Funktionen, *Math. Z.*, **26** (1927), 566–578.

Blecksmith, R.
[1] The determination of Ramanujan pairs, Ph.D. dissertation, University of Arizona, 1983.

Blecksmith, R., Brillhart, J., and Gerst, I.
[1] A computer assisted search for Ramanujan pairs, to appear.

Boas, R. P., Jr.
[1] Partial sums of infinite series, and how they grow, *Amer. Math. Monthly*, **84** (1977), 237–258.
[2] Convergence, divergence, and the computer, in *Mathematical Plums*, R. Honsberger, editor, Mathematical Association of America, Washington, D.C., 1979, chapter 8.

Bochner, S. and Chandrasekharan, K.
[1] *Fourier Transforms*, Princeton University Press, Princeton, 1949.

Briggs, W. E.
[1] Some constants associated with the Riemann zeta-function, *Michigan Math. J.*, **3** (1955–56), 117–121.

Briggs, W. E. and Buschman, R. G.
[1] The power series coefficients of functions defined by Dirichlet series, *Illinois J. Math.*, **5** (1961), 43–44.

Briggs, W. E. and Chowla, S.
[1] The power series coefficients of $\zeta(s)$, *Amer. Math. Monthly*, **62** (1955), 323–325.

Briggs, W. E., Chowla, S., Kempner, A. J., and Mientka, W. E.
[1] On some infinite series, *Scripta Math.*, **21** (1955), 28–30.

Brodén, T. B. N.
[1] Bemerkungen über sogenannte finite Integration, *Arkiv Mat. Astr. och Fys. (Stockholm)*, **7** (1911), 34 pp.
[2] Einige Anwendungen diskontinuierlicher Integrale auf Fragen der Differenzenrechnung, *Acta Univ. Lund.* (2), **8** (1912), 17 pp.

Bromwich, T. J. I'a.
[1] *An Introduction to the Theory of Infinite Series*, 2nd ed., Macmillan, London, 1926.

Browne, D. H.
[1] Problem E461, solution by H. W. Becker, *Amer. Math. Monthly*, **48** (1941), 701–703.

Bruckman, P. S.
[1] Problem H-320, *Fibonacci Quart.*, **20** (1982), 186–187.

Buschman, R. G.
[1] Some infinite series for $\zeta(n+1)$, *Amer. Math. Monthly*, **67** (1960), 260–263.

Büsing, L.
[1] Vergleich von Hauptlösungsbegriffen für Nörlundsche Differenzengleichungen und Anwendungen auf die von Bendersky untersuchten Gamma-funktionen, Dissertation, Technischen Universität Clausthal, 1982.

Carlitz, L.
[1] Congruences for generalized Bell and Stirling numbers, *Duke Math. J.*, **22** (1955), 193–205.

[2] On some polynomials of Tricomi, *Boll. Un. Mat. Ital.* (3), **13** (1958), 58–64.
[3] Eulerian numbers and polynomials, *Math. Mag.*, **33** (1959), 247–260.
[4] Some congruences involving sums of binomial coefficients, *Duke Math. J.*, **27** (1960), 77–79.
[5] Single variable Bell polynomials, *Coll. Math.*, **4** (1962), 13–25.
[6] Eulerian numbers and operators, *Coll. Math.*, **24** (1973), 175–200.
[7] Some remarks on the Eulerian function, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, no. 611 (1978), 79–91.
[8] Some remarks on the Bell numbers, *Fibonacci Quart.*, **18** (1980), 66–73.

Carlitz, L., Kurtz, D. C., Scoville, R., and Stackelberg, O. P.
[1] Asymptotic properties of Eulerian numbers, *Z. Wahrscheinlichkeitstheorie*, **23** (1972), 47–54.

Carlitz, L. and Riordan, J.
[1] Congruences for Eulerian numbers, *Duke Math. J.*, **20** (1953), 339–343.

Carlitz, L. and Scoville, R.
[1] Tangent numbers and operators, *Duke Math. J.*, **39** (1972), 413–429.
[2] The sign of the Bernoulli and Euler numbers, *Amer. Math. Monthly*, **80** (1973), 548–549.
[3] Generalized Eulerian numbers: combinatorial applications, *J. Reine Angew. Math.*, **265** (1974), 110–137; Corrigendum, **288** (1976), 218–219.
[4] Eulerian numbers and operators, *Fibonacci Quart.*, **13** (1975), 71–83.

Carlson, F.
[1] Sur une classe de séries de Taylor, Thesis, Upsala, 1914.

Carmichael, R. D.
[1] On certain transcendental functions defined by a symbolic equation, *Amer. Math. Monthly*, **15** (1908), 78–83.

Carr, G. S.
[1] *Formulas and Theorems in Pure Mathematics*, 2nd ed., Chelsea, New York, 1970.

Catalan, E.
[1] *Mémoire sur la transformation des séries et sur quelques intégrals définies*, Acad. Royale Sci., lettres, et beaux-arts de Belgique, Brussels. Mem. Couronnes savants entrangers, **33** (1865–67), 50 pp.

Cesàro, E.
[1] Sur une équation aux différences mèlées, *Nouv. Ann. Math.* (3), **4** (1885), 36–40.

Chaundy, T.
[1] *The Differential Calculus*, Clarendon Press, Oxford, 1935.

Chrystal, G.
[1] *Algebra*, part II, 2nd ed., A. and C. Black, London, 1922.

Clausen, T.
[1] Addendum to "Beitrag zur Theorie der Reihen," *J. Reine Angew. Math.*, **3** (1828), 95.
[2] Über die Summe der Reihen $1 + \frac{1}{2}^3 + \frac{1}{3}^3 + \frac{1}{4}^4 \ldots$ und $1 + \frac{1}{7}^2 + \frac{1}{13}^2 + \frac{1}{19}^2 + \ldots$, *J. Reine Angew. Math.*, **5** (1830), 380–382.

Cohen, H.
[1] Démonstration de l'irrationalité de $\zeta(3)$ (d'après R. Apéry), *Sém. Théorie des Nombres*, Grenoble, 1978, 9 pp.
[2] Généralisation d'une construction de R. Apéry, *Bull. Soc. Math. France*, **109**, (1981), 269–281.

Comtet, L.
[1] *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

Creutz, M. and Sternheimer, R. M.

[1] On the convergence of iterated exponentials — I, *Fibonacci Quart.*, **18** (1980), 341–347.

Davenport, H.

[1] *Multiplicative Number Theory*, 2nd ed., Springer-Verlag, New York, 1980.

Deninger, C.

[1] On the analogue of the formula of Chowla and Selberg for real quadratic fields, *J. Reine Angew. Math.*, **351** (1984), 171–191.

Djoković, D. Ž.

[1] Summation of certain types of series, *Publ. Math. Inst. (Beograd)*, no. 18, **4** (1964), 43–55.

Dobiński, G.

[1] Summirung der Reihe $\sum n^m/n!$ für $m = 1, 2, 3, 4, 5, \ldots$, *Arch. Math. Phys.*, **61** (1877), 333–336.

Duparc, J. A., Lekkerkerker, C. G., and Peremans, W.

[1] An elementary proof of a formula of Jensen, *Math. Centrum Amsterdam*, Rapport ZW 1952-021, 1952.

Dwight, H. B.

[1] *Mathematical Tables of Elementary and Some Higher Mathematical Functions Including Trigonometric Functions of Decimals of Degrees and Logarithms*, McGraw-Hill, New York, 1941.

Edwards, H. M.

[1] *Riemann's Zeta-function*, Academic Press, New York, 1974.

Edwards, J.

[1] *Differential Calculus*, 2nd ed., Macmillan, London, 1982.

[2] *A Treatise on the Integral Calculus*, 2 vols, Macmillan, London, 1922; reprinted by Chelsea, New York, 1954.

Eisenstein, G.

[1] Entwicklung von $\alpha^{\alpha^{\alpha^{\cdot^{\cdot^{\cdot}}}}}$, *J. Reine Angew. Math.*, **28** (1844), 49–52.

[2] *Mathematische Werke*, vol. 1, Chelsea, New York, 1975.

Epstein, L. F.

[1] A function related to the series for e^{e^x}, *J. Math. Phys. Mass. Inst. Tech.*, **18** (1939), 153–173.

Erdélyi, A., editor

[1] *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New York, 1953.

Euler, L.

[1] *Institutiones Calculi Differentialis*, Acad. Imperialis Sci., Petrograd, 1755.

[2] Meditationes circa singulare serierum genus, *Novi. Comm. Acad. Sci. Petropolitanae*, **20** (1775), 140–186.

[3] De formulis exponentialibus replicatis, *Acta Acad. Sci. Petropolitanae*, **1** (1778), 38–60.

[4] De serie Lambertina plurimisque eius insignibus proprietatibus, *Acta Acad. Sci. Petropolitanae* 1779: **II** (1783), 29–51.

[5] De summatione serierum in hac forma contenarum $a/1 + a^2/4 + a^3/9 + a^4/16 + a^5/25 + a^6/36 +$ etc., *Mem. de l'Acad. de St. Pétersbourg*, **3** (1809/10), 26–42.

[6] *Opera Omnia*, ser. 1, vol. 6, B. G. Teubner, Leipzig, 1921.

[7] *Opera Omnia*, ser. 1, vol. 10, B. G. Teubner, Leipzig, 1913.

[8] *Opera Omnia*, ser. 1, vol. 15, B. G. Teubner, Berlin, 1927.

[9] *Opera Omnia*, ser. 1, vol. 16, part 2, B. G. Teubner, Berlin, 1935.

[10] *Briefwechsel*, vol. 1, Birkhäuser, Basel, 1975.

Euler, L. and Goldbach, C.
[1] *Briefwechsel* 1729–1764, Akademie-Verlag, Berlin, 1965.
Faà di Bruno, F.
[1] *Einleitung in die Theorie der binären Formen*, trans. by M. Noether, B. G. Teubner, Leipzig, 1881.
Fee, G.
[1] A computation of Catalan's constant (preprint).
Feller, W.
[1] *An Introduction to Probability Theory and Its Applications*, vol. 2, Wiley, New York, 1966.
Ferguson, R. P.
[1] An application of Stieltjes integration to the power series coefficients of the Riemann zeta-function, *Amer. Math. Monthly*, **70** (1963), 60–61.
Ferrero, B. and Greenberg, R.
[1] On the behavior of p-adic L-functions at $s = 0$, *Invent. Math.*, **50** (1978), 91–102.
Fichtenholz, G. M.
[1] *Differential und Integralrechnung*, Band 2, zweite Auflage, VEB Deutscher Verlag der Wissenschaften, Berlin, 1966.
Fletcher, A., Miller, J. C. P., Rosenhead, L., and Comrie, L. J.
[1] *An Index of Mathematical Tables*, vol. I, 2nd ed., Addison-Wesley, Reading, Massachusetts, 1962.
Francis, E. C. and Littlewood, J. E.
[1] *Examples in Infinite Series with Solutions*, Deighton Bell, Cambridge, 1928.
Fransén, A. and Wrigge, S.
[1] High-precision values of the gamma function and of some related coefficients, *Math. Comp.*, **34** (1980), 553–566.
Frobenius, F. G.
[1] Über die Bernoullischen Zahlen und die Eulerschen Polynome, Sitz. d. K. Preuss. Akad. Wiss., Berlin 1910, pp. 809–847.
[2] *Gesammelte Abhandlungen*, Band III, Springer-Verlag, Berlin, 1968.
Gastmans, R. and Troost, W.
[1] On the evaluation of polylogarithmic integrals, *Simon Stevin*, **55** (1981), 205–219.
Gates, D. J., Gerst, I., and Kac, M.
[1] Non-Markovian diffusion in idealized Lorentz gases, *Arch. Rational Mech. Anal.*, **51** (1973), 106–135.
Gates, L. W.
[1] Summing square roots, *Math. Gazette*, **64** (1980), 86–89.
Gauss, C. F.
[1] *Disquisitiones Arithmeticae*, trans. by A. A. Clarke, Yale University Press, New Haven, 1966.
Georghiou, C. and Philippou, A. N.
[1] Harmonic sums and the zeta-function, *Fibonacci Quart.*, **21** (1983), 29–36.
Ginsburg, J.
[1] Iterated exponentials, *Scripta Math.*, **11** (1945), 340–353.
Glaisher, J. W. L.
[1] On the constants that occur in certain summations by Bernoulli's series, *Proc. London Math. Soc.*, **4** (1872), 48–56.
[2] A theorem in trigonometry, *Quart. J. Math.*, **15** (1878), 151–157.
[3] Summation of certain numerical series, *Mess. Math.*, **42** (1913), 19–34.
Glasser, M. L. and Klamkin, M. S.
[1] On some inverse tangent summations, *Fibonacci Quart.*, **14** (1976), 385–388.

Gosper, R. W., Jr.
[1] A calculus of series rearrangements, in *Algorithms and Complexity*, J. F. Traub, editor, Academic Press, New York, 1976, pp. 121–151.

Gould, H. W.
[1] Some generalizations of Vandermonde's convolution, *Amer. Math. Monthly*, **63** (1956), 84–91.
[2] Final analysis of Vandermonde's convolution, *Amer. Math. Monthly*, **64** (1957), 409–415.
[3] Generalization of a theorem of Jensen concerning convolutions, *Duke Math. J.*, **27** (1960), 71–76.
[4] A series transformation for finding convolution identities, *Duke Math. J.*, **28** (1961), 193–202.
[5] A new convolution formula and some new orthogonal relations for the inversion of series, *Duke Math. J.*, **29** (1962), 393–404.
[6] Congruences involving sums of binomial coefficients and a formula of Jensen, *Amer. Math. Monthly*, **69** (1962), 400–402.
[7] Brief guide to combinatorial addition theorems (unpublished), 1968.
[8] *Combinatorial Identities*, Morgantown Printing and Binding Co., Morgantown, 1972.
[9] *Bell & Catalan Numbers*, Combinatorial Research Institute, Morgantown, 1976.

Gradshteyn, I. S. and Ryzhik, I. M.
[1] *Tables of Integrals, Series, and Products*, 4th ed. Academic Press, New York, 1965.

Gram, J. P.
[1] Note sur le calcul de la fonction $\zeta(s)$ de Riemann, *Oversigt. K. Danske Vidensk. Selskab. Forh.* (1895), 305–308.
[2] Tafeln für die Riemannsche Zetafunktion, *K. Danske Vidensk. Selskab. Skr.* (8), **10** (1925), 313–325.

Greenstreet, W. J.
[1] Summation of certain infinite series, *Amer. Math. Monthly*, **14** (1907), 41–47.

Gross, B. H. and Koblitz, N.
[1] Gauss sums and the p-adic Γ-function, *Ann. Math.*, **109** (1979), 569–581.

Guinand, A. P.
[1] A note on the logarithmic derivative of the gamma function, *Edinburgh Math. Notes*, **38** (1952), 1–4.

Gupta, H.
[1] An identity, *Res. Bull. Panjab Univ.* (N.S.), **15** (1964–65), 347–349.

Hadamard. J. S. and Mandelbrojt, M.
[1] *La Série de Taylor*, Gauthier-Villars, Paris, 1926.

Hans, R. J. and Dumir, V. C.
[1] An interesting identity, *Res. Bull. Panjab Univ.* (N.S.), **15** (1964–65), 353–356.

Hansen, E. R.
[1] *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

Hansen, E. R. and Patrick, M. L.
[1] Some relations and values for the generalized Riemann zeta-function, *Math. Comp.*, **16** (1962), 265–274; Corrigenda, **17** (1963), 104.

Hardy, G. H.
[1] On the Frullanian integral $\int_0^\infty\{[\varphi(ax^m)-\psi(bx^n)]/x\}\ (\log x)^p\,dx$, *Quart. J. Math.*, **33** (1902), 113–144.
[2] A generalization of Frullani's integral, *Mess. Math.*, **34** (1905), 11–18.

[3] Note on a former paper, *Mess. Math.*, **34** (1905), 102.
[4] On the zeros of certain classes of integral Taylor series. Part II—on the integral function $\sum_{n=0}^{\infty}(x^n/(n+a)^s n!)$ and other similar functions, *Proc. London Math. Soc.* (2), **2** (1905), 401–431.
[5] On the expression of the double zeta-function and double gamma-function in terms of elliptic functions, *Trans. Cambridge Phil. Soc.*, **20** (1905), 1–35.
[6] *Orders of Infinity*, Cambridge University Press, London, 1910.
[7] Note on Dr. Vacca's series for γ, *Quart. J. Pure Appl. Math.*, **43** (1912), 215–216.
[8] Proof of a formula of Mr. Ramanujan, *Mess. Math.*, **44** (1915), 18–21.
[9] Srinivasa Ramanujan, *Proc. London Math. Soc.* (2), **19** (1921), xl–lviii.
[10] Srinivasa Ramanujan, *Proc. Royal Soc. London* A,, **99** (1921), xiii–xxix.
[11] Srinivasa Ramanujan, *J. Indian Math. Soc.*, **14** (1922), 82–104.
[12] A chapter from Ramanujan's note-book, *Proc. Cambridge Phil. Soc.*, **21** (1923), 492–503.
[13] Ramanujan and the theory of transforms, *Quart. J. Math.*, **8** (1937), 245–254.
[14] The Indian mathematician Ramanujan, *Amer. Math. Monthly*, **44** (1937), 137–155.
[15] *Divergent Series*, Clarendon Press, Oxford, 1949.
[16] *A Course of Pure Mathematics*, 10th ed., Cambridge University Press, Cambridge, 1967.
[17] *A Mathematician's Apology*, Cambridge University Press, Cambridge, 1967.
[18] *Collected Papers*, vol. IV, Clarendon Press, Oxford, 1969.
[19] *Collected Papers*, vol. V, Clarendon Press, Oxford, 1972.
[20] *Ramanujan*, 3rd ed., Chelsea, New York, 1978.
[21] *Collected Papers*, vol. VII, Clarendon Press, Oxford, 1979.

Hardy, G. H. and Wright, E. M.
[1] *An Introduction to the Theory of Numbers*, 4th ed., Clarendon Press, Oxford, 1960.

Henrici, P.
[1] *Applied and Computational Complex Analysis*, vol. 1, Wiley, New York, 1974.
[2] *Applied and Computational Complex Analysis*, vol. 2, Wiley, New York, 1977.

Hill, J. M., Laird, P. G., and Cerone, P.
[1] Mellin-type integral equations for solutions of differential-difference equations, *Utilitas Math.*, **15** (1979), 129–141.

Hirschhorn, M. D.
[1] Two further Ramanujan pairs, *J. Austral. Math. Soc.*, **30** (1980), 1–4.

Hjortnaes, M. M.
[1] Overføring av rekken $\sum_{k=1}^{\infty} 1/k^3$ til et bestemt integral, in *Tolfte Skandinaviska matematikerkongressen*, Lund, 10–15 Augusti 1953, H. Ohlssons Boktr., Lund, 1954.

Howard, F. T.
[1] Explicit formulas for numbers of Ramanujan, *Fibonacci Quart.*, to appear.

Ishibashi, M. and Kanemitsu, S.
[1] Fractional part sums and divisor functions, I, in preparation.

Israilov, M. I.
[1] On the Hurwitz zeta function, *Izv. Akad. Nauk Uz SSR, Ser. Fiz.-Mat. Nauk* 1981, no. 6 (1981), 13–18 (Russian).

Jackson, F. H.
[1] A q-generalization of Abel's series, *Rend. Circ. Mat. Palermo*, **29** (1910), 340–346.

Jensen, J. L. W. V.
[1] Sur la fonction $\zeta(s)$ de Riemann, *C.R. Acad. Sci. (Paris)*, **104** (1887), 1156–1159.

[2] Sur une identité d'Abel et sur d'autres formules analogues, *Acta Math.*, **26** (1902), 307–318.

Jordan, P. F.

[1] A reversible transformation and related sets of Legendre coefficients, AFOSR Scientific Report, Martin Marietta Corp., Baltimore, 1972.

[2] Infinite sums of psi functions, *Bull. Amer. Math. Soc.*, **79** (1973), 681–683.

Katznelson, Y.

[1] *An Introduction to Harmonic Analysis*, 2nd ed., Dover, New York, 1976.

Kesava Menon, P.

[1] On a function of Ramanujan, *J. Indian Math. Soc.*, **25** (1961), 109–119.

[2] Summation of certain series, *J. Indian Math. Soc.*, **25** (1961), 121–128.

Klamkin, M. S.

[1] Problem 4431, sol. by R. Steinberg, *Amer. Math. Monthly*, **59** (1952), 471–472.

[2] Problem 4564, sols. by J. V. Whittaker and the proposer, *Amer. Math. Monthly*, **62** (1955), 129–130.

Kluyver, J. C.

[1] On certain series of Mr. Hardy, *Quart. J. Pure Appl. Math.*, **50** (1927), 185–192.

Knoebel, R. A.

[1] Exponentials reiterated, *Amer. Math. Monthly*, **88** (1981), 235–252.

Knopfmacher, J.

[1] Generalized Euler constants, *Proc. Edinburgh Math. Soc.*, **21** (1978), 25–32.

Knopp, K.

[1] *Theory and Application of Infinite Series*, 2nd ed., Blackie & Son, Glasgow, 1951.

Knuth, D. E.

[1] Evaluation of Porter's constant, *Comp. Math. Appl.*, **2** (1976), 137–139.

Kramp, C.

[1] Coefficient des allgemeinen Gliedes jeder willkührlichen Potenz eines Infinitinomiums; Verhalten zwischen Coefficienten der Gleichungen und Summen der Produkte und der Potenzen ihrer Wurzeln; Transformation und Substitution der Reihen durch einander, im *Sammlung combinatorischanalytischer Abhandlungen*, C. F. Hindenburg, editor, Gerhard Fleischer, Leipzig, 1796, pp. 91–122.

Kummer, E. E.

[1] Beitrag zur Theorie der Function $\Gamma(x)=\int_0^\infty e^{-v}v^{x-1}\,dv$, *J. Reine Angew. Math.*, **35** (1847), 1–4.

Lagrange, J. L.

[1] Nouvelle méthode pour résondre les équations littérales par le moyen des séries, in *Oeuvres*, vol. 3, Gauthier-Villars, Paris, 1869, pp. 5–73.

Lambert, J. H.

[1] Observationes variae in mathesin puram, in *Opera Mathematica*, vol. 1, Orell Fussli, Zurich, 1946, pp. 16–51.

Lammel, E.

[1] Ein Beweis, dass die Riemannsche Zetafunktion $\zeta(s)$ in $|s-1|\leq 1$ keine Nullstelle besitzt, *Univ. nac Tucumán, Revista*, Ser. A, **16** (1966), 209–217.

Landau, E.

[1] *Primzahlen*, Chelsea, New York, 1953.

Landen, J.

[1] *Mathematical Memoirs*, J. Nourse, London, 1780.

Lavrik, A. F.

[1] Laurent coefficients of a generalized zeta-function, in *Theory of Cubature*

Formulas and Numerical Mathematics (Prof. Conf., Novosibirsk, 1978), "Nauka" Sibirsk. Otdel., Novosibirsk, 1980, pp. 160–164, 254 (Russian).

Legendre, A. M.

[1] *Traité des Fonctions Elliptiques et des Intégrales Eulériennes*, tome II, Huzard-Courcier, Paris, 1826.

Lense, J.

[1] Problem 3053, *Amer. Math. Monthly*, **31** (1924), 500–501.

Lerch, M.

[1] Généralisation du théorème de Frullani, *Sitz. Kgl. Böhm. Gesell. Wiss. Math.-Natur.* Cl. 1893, no. 30, 6 pp.

[2] Dalši studie v oboru Malmsténovských řad, *Rozpravy Ceske Akad.*, no. 28, **3** (1894), 63 pp.

Leshchiner, D.

[1] Some new identities for $\zeta(k)$, *J. Number Theory*, **13** (1981), 355–362.

Levine, J. and Dalton, R. E.

[1] Minimum periods, modulo p, of first-order Bell exponential integers, *Math. Comp.*, **16** (1962), 416–423.

Lewin, L.

[1] *Polylogarithms and Associated Functions*, North-Holland, New York, 1981.

[2] The dilogarithm in algebraic fields, *J. Austral. Math. Soc.*, Ser. A, **33** (1982), 302–330.

Liang, J. J. Y. and Todd, J.

[1] The Stieltjes constants, *J. Res. National Bureau of Standards, Math. Sci.* **B**, **76** (1972), 161–178.

Lionnet, M.

[1] Question 1294, solution by M. A. Laisant, *Nouv. Ann. Math.*, ser. 2, **18** (1879), 330–332.

Littlewood, J. E.

[1] Review of "Collected Papers of Srinivasa Ramanujan," *Math. Gaz.*, **14** (1929), 425–428.

[2] *Collected Papers*, vol. 2, Clarendon Press, Oxford, 1982.

Loney, S. L.

[1] *Plane Trigonometry*, parts I, II, Cambridge University Press, Cambridge, 1893.

Loxton, J. H.

[1] Special values of the dilogarithm function, *Acta Arith.*, **43** (1984), 155–166.

MacLeod, R. A.

[1] Fractional part sums and divisor functions, *J. Number Theory*, **14** (1982), 185–227.

Manikarnikamma, S. N.

[1] Some properties of the series $\sum_{n=0}^{\infty} (n+a)^r x^n/n!$, *Math. Student*, **18** (1950), 132–135.

Matsuoka, Y.

[1] On the values of a certain Dirichlet series at rational integers, *Tokyo J. Math.*, **5** (1982), 399–403.

Mendès France, M.

[1] Roger Apéry et l'irrationnel, *La Recherche*, **10** (1979), 170–172.

Milnor, J.

[1] On polylogarithms, Hurwitz zeta functions, and the Kubert identities, *L'Enseign. Math.*, **29** (1983), 281–322.

Mitrović, D.

[1] The signs of some constants associated with the Riemann zeta-function, *Michigan Math. J.*, **9** (1962), 395–397.

Mordell, L. J.
[1] Ramanujan, *Nature*, **148** (1941), 642–647.
Moreno, C. J.
[1] The Chowla–Selberg formula, *J. Number Theory*, **17** (1983), 226–245.
Moy, A.
[1] Problem 2723, sol. by R. Breusch, *Amer. Math. Monthly*, **86** (1979), 788–789.
Nandy, A.
[1] *Alternative Sciences*, Allied Publishers, New Delhi, 1980.
Nandy, P.
[1] Conversation (interview with S. Janaki Ammal).
Neville, E. H.
[1] Srinivasa Ramanujan, *Nature*, **149** (1942), 292–295.
Newman, D. J.
[1] Problem 4489, sols, by H. F. Sandham, R. Frucht, and M. S. Klamkin, *Amer. Math. Monthly*, **60** (1953), 484–485.
Nielsen, N.
[1] Undersøgelser over reciproke potensummer og deres anvendelse paa raekker og integraler, *K. Danske Vids.* Selsk. Skr. Natur. Math. (6), **8** (1898), 395–443.
[2] Recherches sur le carré da la dérivée logarithmique de la fonction gamma et sur quelques fonctions analogues, *Annali Math.*, **9** (1904), 189–210.
[3] Recherches sur des généralisations d'une fonction de Legendre et d'Abel, *Annali Math.*, **9** (1904), 219–235.
[4] Der Eulersche Dilogarithmus und seine Verallgemeinerungen, *Nova Acta, Abh. der Kaiserl. Leopoldinisch-Carolinischen Deutschen Akad. der Naturforscher*, **90** (1909), 121–212.
[5] *Traité Élémentaire des Nombres de Bernoulli*, Gauthier-Villars, Paris, 1923.
[6] *Handbuch der Theorie der Gamma-funktion*, Chelsea, New York, 1965.
Nörlund, N. E.
[1] IIC7. Neure Untersuchungen über Differenzengleichungen, in *Encyklopädie der mathematischen Wissenschaften*, Band 2, Teil 3, B. G. Teubner, Leipzig, 1923, pp. 675–721.
[2] *Vorlesungen über Differenzenrechnung*, Chelsea, New York, 1954.
Ogilvy, C. S.
[1] Elementary problem E853, sol. by P. G. Kirmser, *Amer. Math. Monthly*, **56** (1949), 555–556.
Oldham, K. B. and Spanier, J.
[1] *The Fractional Calculus*, Academic Press, New York, 1974.
Olver, F. W. J.
[1] *Asymptotics and Special Functions*, Academic Press, New York, 1974.
Ostrowski, A. M.
[1] On some generalizations of the Cauchy–Frullani integral, *Proc. Nat. Acad. Sci.*, **35** (1949), 612–616.
[2] On Cauchy–Frullani integrals, *Comment. Math. Helvetici*, **51** (1976), 57–91.
Picard, E.
[1] Sur une classe de transcendantes nouvelles, *Acta Math.*, **18** (1894), 133–154.
Pollak, H. O. and Shepp, L.
[1] Problem 64–1, an asymptotic expansion, solution by J. H. Van Lint, *SIAM Review*, **8** (1966), 383–384.
Pólya, G. and Szegö, G.
[1] *Problems and Theorems in Analysis*, vol. 1, Springer-Verlag, Berlin, 1972.
Post, E. L.
[1] The generalized gamma function, *Ann. Math.* (2), **20** (1919), 202–217.

Problem
[1] *Math. Sb.*, **4** (1869), 39.

Ram, S.
[1] *Srinivasa Ramanujan*, National Book Trust, New Delhi, 1972.

Ramachandra Rao, R.
[1] In memoriam: S. Ramanujan, *J. Indian Math. Soc.*, **12** (1920), 87–90.

Ramanathan, K. G.
[1] The unpublished manuscripts of Srinivasa Ramanujan, *Current Sci.*, **50** (1981), 203–210.

Ramanujan, S.
[1] Question 260, *J. Indian Math. Soc.*, **3** (1911), 43.
[2] Question 261, *J. Indian Math. Soc.*, **3** (1911), 43.
[3] Question 327, *J. Indian Math. Soc.*, **3** (1911), 209.
[4] Some properties of Bernoulli's numbers, *J. Indian Math. Soc.*, **3** (1911), 219–234.
[5] On question 330 of Prof. Sanjana, *J. Indian Math. Soc.*, **4** (1912), 59–61.
[6] Question 386, *J. Indian Math. Soc.*, **4** (1912), 120.
[7] Irregular numbers, *J. Indian Math. Soc.*, **5** (1913), 105–106.
[8] Question 606, *J. Indian Math. Soc.*, **6** (1914), 239.
[9] Question 642, *J. Indian Math. Soc.*, **7** (1915), 80.
[10] On the integral $\int_0^x (\tan^{-1} t)/t \, dt$, *J. Indian Math. Soc.*, **7** (1915), 93–96.
[11] On the sum of the square roots of the first n natural numbers, *J. Indian Math. Soc.*, **7** (1915), 173–175.
[12] On the product $\prod_{n=0}^{n=\infty}[1 + (x/(a + nd))^3]$, *J. Indian Math. Soc.*, **7** (1915), 209–211.
[13] Some definite integrals, *Mess. Math.*, **44** (1915), 10–18.
[14] Some formulae in the analytic theory of numbers, *Mess. Math.*, **45** (1916), 81–84.
[15] *Collected Papers*, Chelsea, New York, 1962.
[16] *Notebooks*, 2 vols., Tata Institute of Fundamental Research, Bombay, 1957.

Ranganathan, S. R.
[1] *Ramanujan: The Man and the Mathematician*, Asia Publishing House, Bombay, 1967.

Rankin, R. A.
[1] Ramanujan's manuscripts and notebooks, *Bull. London Math. Soc.*, **14** (1982), 81–97.
[2] Ramanujan as a patient, *J. Indian Math. Soc.*, to appear.

Reyssat, E.
[1] Irrationalité de $\zeta(3)$ selon Apéry, *Sém. Delange–Pisot–Poitou*, **20** (1978/79), 6 pp.

Richmond, B. and Szekeres, G.
[1] Some formulas related to dilogarithms, the zeta-function and the Andrews–Gorden identities, *J. Austral. Math. Soc.*, series A, **31** (1981), 362–373.

Riordan, J.
[1] *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.
[2] *Combinatorial Identities*, Wiley, New York, 1968.

Robinson, M. L.
[1] On a function of Ramanujan, in preparation.

Rogers, K.
[1] Solving an exponential equation, *Math. Mag.*, **53** (1980), 26–28.

Roman, S. M. and Rota, G.-C.
[1] The umbral calculus, *Adv. Math.*, **27** (1978), 95–188.

Ross, B., editor
[1] *Fractional Calculus and Its Applications*, Springer-Verlag, Berlin, 1975.
[2] The development of fractional calculus 1695–1900, *Hist. Math.*, 4 (1977), 75–89.

Rosser, J. B. and Schoenfeld, L.
[1] Approximate formulas for some functions of prime numbers, *Illinois J. Math.*, **6** (1962), 64–94.

Rota, G.-C., Kahaner, D., and Odlyzko, A.
[1] On the foundations of combinatorial theory. VIII. Finite operator calculus, *J. Math. Anal. Appl.*, **42** (1973), 684–760.

Rota, G.-C. and Mullin, R.
[1] On the foundations of combinatorial theory, in *Graph Theory and Its Applications*, B. Harris, editor, Academic Press, New York, 1970, pp. 167–213.

Rothe, H. A.
[1] Formulae de serierum reversione demonstratio universalis signis localibus combinatorioanalyticorum vicariis exhibita, Dissertation, Leipzig, 1793.

Ruscheweyh, S.
[1] Einige neue Darstellungen des Di und Trilogarithmus, *Math. Nachr.*, **57** (1973), 237–244.

Rutledge, G. and Douglass, R. D.
[1] Evaluation of $\int_0^1 \{(\log u)/u\} \log^2(1+u)\,du$, *Amer. Math. Monthly*, **41** (1934), 29–36.

Schaeffer, S.
[1] De integrali $-\int_0^x \{(\log(1-\alpha))/\alpha\}\partial\alpha$, *J. Reine Angew. Math.*, **30** (1844), 277–295.

Scherk, H. F.
[1] Bemerkungen über die Lambertsche Reihe

$$\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^3}+\frac{x^4}{1-x^4}+\text{etc.},$$

J. Reine Angew. Math., **9** (1832), 162–168.

Schlömilch, O.
[1] *Compendium der höheren Analysis in zwei Bänden*, fünfte Auflage, Friedrich Vieweg und Sohn, Braunschweig, 1881.

Seidel, L.
[1] Ueber die Grenzwerthe eines unendlichen Potenzausdruckes der Form $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, *Abh. K. Bayer. Akad. Wiss. Math.-Phys. Kl.*, **11** (1873), 1–10.

Shell, D. L.
[1] On the convergence of infinite exponentials, *Proc. Amer. Math. Soc.*, **13** (1962), 678–681.

Shintani, T.
[1] On special values of zeta-functions of totally real quadratic fields, in *Proceedings of the International Congress of Mathematicians*, Helsinki, 1978, vol. 2, Acad. Sci. Fennica, Helsinki, 1980, pp. 591–597.

Sitaramachandrarao, R. and Siva Rama Sarma, A.
[1] Some identities involving the Riemann zeta-function, *Indian J. Pure Appl. Math.*, **10** (1979), 602–607.
[2] Two identities due to Ramanujan, *Indian J. Pure Appl. Math.*, **11** (1980), 1139–1140.

Sitaramachandrarao, R. and Subbarao, M. V.
[1] On some infinite series of L. J. Mordell and their analogues, *Pacific J. Math.*, to appear.
[2] Transformation formulae for multiple series, *Pacific J. Math.*, **113** (1984), 471–479.

Siva Rama Sarma, A.
[1] Some problems in the theory of Farey series and the Euler totient function, Ph.D. thesis, University of Waltair, 1981.

Snow, C. P.
[1] *Variety of Men*, Charles Scribner's Sons, New York, 1966.

Snow, D. R.
[1] Formulas for sums of powers of integers by functional equations, *Aequa. Math.*, **18** (1978), 269–285.

Spence, W.
[1] *An Essay on Logarithmic Transcendents*, Murray, London, 1809.

Srinivasan, P. K., editor
[1] *Ramanujan: An Inspiration*, memorial vols., 1 and 2, Muthialpet High School, Madras, 1968.

Stanley, R. P.
[1] Generating functions, in *Studies in Combinatorics*, G.-C. Rota, editor, Mathematical Association of America, Washington, D.C., 1978, pp. 100–141.

Stark, H. M.
[1] *An Introduction to Number Theory*, Markham, Chicago, 1970.
[2] Values of zeta- and *L*-functions, *Abh. Braunschweig Wiss. Ges.*, **33** (1982), 71–83.

Stieltjes, T. J.
[1] Table des valeurs des sommes $S_k = \sum_1^\infty n^{-k}$, *Acta Math.*, **10** (1887), 299–302.
[2] *Correspondance d'Hermite et de Stieltjes*, Tome 1, Gauthier-Villars, Paris, 1905.

Tate, J.
[1] *A Treatise on Factorial Analysis*, Bell, London, 1845.

Titchmarsh, E. C.
[1] *The Theory of Functions*, 2nd ed., Oxford University Press, London, 1939.
[2] *Introduction to the Theory of Fourier Integrals*, 2nd ed., Clarendon Press, Oxford, 1948.
[3] *The Theory of the Riemann Zeta-Function*, Clarendon Press, Oxford, 1951.

Touchard, J.
[1] Propriétés arithmétiques de certains nombres récurrents, *Ann. Soc. Sci. Bruxelles* A, **53** (1933), 21–31.
[2] Nombres exponentiels et nombres de Bernoulli, *Canad. J. Math.*, **8** (1956), 305–320.

Turnbull, H. W.
[1] Bertram Martin Wilson, M.A., D.Sc., *Proc. Royal Soc. Edinburgh*, **55** (1936), 176–177.

Uspensky, J. V. and Heaslet, M. A.
[1] *Elementary Number Theory*, McGraw-Hill, New York, 1939.

van der Poorten, A.
[1] A proof that Euler missed, ... Apéry's proof of the irrationality of $\zeta(3)$, *Math. Intell.*, **1** (1979), 195–203.
[2] Some wonderful formulae ... footnotes to Apéry's proof of the irrationality of $\zeta(3)$, *Sém. Delange–Pisot–Poitou*, **20** (1978/79), 7 pp.
[3] Some wonderful formulas ... an introduction to polylogarithms, in *Proceed-*

ings of Queen's Number Theory Conference, 1979, P. Ribenboim, editor, Queen's Papers in Pure and Applied Mathematics, No. 54, Kingston, 1980, pp. 269–286.

Verma, D. P.
[1] Laurent's expansion of Riemann's zeta-function, *Indian J. Math.*, **5** (1963), 13–16.

Wagstaff, S. S., Jr.
[1] The irregular primes to 125000, *Math. Comp.*, **32** (1978), 583–591.
[2] Ramanujan's paper on Bernoulli numbers, *J. Indian Math. Soc.*, **45** (1981), 49–65.

Wall, H. S.
[1] *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948.

Watson, G. N.
[1] Theorems stated by Ramanujan (VIII): Theorems on divergent series, *J. London Math. Soc.*, **4** (1929), 82–86.
[2] Ramanujan's notebooks, *J. London Math. Soc.*, **6** (1931), 137–153.
[3] *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1966.

Wheelon, A. D.
[1] *Tables of Summable Series and Integrals Involving Bessel Functions*, Holden-Day, San Francisco, 1968.

Whittaker, E. T. and Robinson, G.
[1] *The Calculus of Observations*, 2nd ed., Blackie and Son, Glasgow, 1926.

Whittaker, E. T. and Watson, G. N.
[1] *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1966.

Williams, G. T.
[1] Numbers generated by the function e^{e^x-1}, *Amer. Math. Monthly*, **52** (1945), 323–327.
[2] A new method of evaluating $\zeta(2n)$, *Amer. Math. Monthly*, **60** (1953), 19–25.

Williamson, B.
[1] *An Elementary Treatise on the Integral Calculus*, 7th ed., Longmans, Green, and Co., New York, 1896.

Wilton, J. R.
[1] A note on the coefficients in the expansion of $\zeta(s, x)$ in powers of $s-1$, *Quart. J. Pure Appl. Math.*, **50** (1927), 329–332.

Wittstein, L.
[1] Auflösung der Gleichung $x^y = y^x$ in reellen Zahlen, *Arch. Math. Phys.*, **6** (1845), 154–163.

Woepcke, F.
[1] Note sur l'expression $(((a^a)^a)^{\cdot^{\cdot^{\cdot^{a}}}})^a$ et les fonctions inverses correspondantes, *J. Reine Angew. Math.*, **42** (1851), 83–90.

Worpitzky, J.
[1] Studien über die Bernoullischen und Eulerschen Zahlen, *J. Reine Angew. Math.*, **94** (1883), 203–232.

Wrench, J. W., Jr.
[1] Review of "Tables of the Riemann zeta function and related functions" by A. McLellan IV, *Math. Comp.*, **22** (1968), 687–688.
[2] Personal communication, October 27, 1980.

Wright, E. M.
[1] Solution of the equation $ze^z = a$, *Bull. Amer. Math. Soc.*, **65** (1959), 89–93.
[2] Solution of the equation $ze^z = a$, *Proc. Royal Soc. Edinburgh*, Sect. A, **65** (1959), 192–203.
[3] Solution of the equation $(pz + q)e^z = rz + s$, *Bull. Amer. Math. Soc.*, **66** (1960), 277–281.
[4] Solution of the equation $(z + b)e^{z+a} = \pm(z + b)$, *Proc. Royal Soc. Edinburgh*, Sect. A, **65** (1960/61), 358–371.
[5] Stability criteria and the real roots of a transcendental equation, *SIAM J. Appl. Math.*, **9** (1961), 136–148.

Zhang, N. Y.
[1] On the Stieltjes constants of the zeta function (Chinese), *Beijing Daxue Xuebao*, no. 4, (1981), 20–24.

Zucker, I. J.
[1] On the series $\sum_{k=1}^{\infty}\binom{2k}{k}^{-1} k^{-n}$ and related sums, *J. Number Theory*, to appear.

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